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Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité. EULER

. . . *ut proinde his paucis consideratis tota haec materia redacta sit ad puram Geometriam, quod in physicis & mechanicis unice desideratum.* LEIBNIZ

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On Two-Dimensional Variational Problems in Parametric Form

H. B. JENKINS

Communicated by R. FINN

I. Introduction. We shall consider two-dimensional variational problems of the type

(1) $$\delta \int \mathscr{J}(\boldsymbol{p})\, ds\, dt = 0.$$

The integral applies to surfaces given in parametric form by functions $x(s, t)$, $y(s, t)$, $z(s, t)$, and $\boldsymbol{p} = (\sigma, \tau, \omega)$ where $\sigma = \partial(y, z)/\partial(s, t)$, $\tau = \partial(z, x)/\partial(s, t)$, $\omega = \partial(x, y)/\partial(s, t)$. We shall assume that $\mathscr{J}(\boldsymbol{p})$ belongs to the Hölder class $C^{2+\alpha}$ for all $\boldsymbol{p} \neq 0$, and that (i) $\mathscr{J}(\boldsymbol{p})$ is positive homogeneous of degree 1; *i.e.*, $\mathscr{J}(\lambda \boldsymbol{p}) = \lambda \mathscr{J}(\boldsymbol{p})$ for all $\lambda > 0$; and (ii) $\mathscr{J}(\boldsymbol{p})$ is positive definite; *i.e.*, there exist positive constants m_1, m_2 such that $m_1 |\boldsymbol{p}| \leqq \mathscr{J}(\boldsymbol{p}) \leqq m_2 |\boldsymbol{p}|$. For applications we shall assume in addition the condition (iii) $\mathscr{J}(\boldsymbol{p})$ is regular. We define $\mathscr{J}(\boldsymbol{p})$ to be regular if the surface Σ defined by $\mathscr{J}(\boldsymbol{p}) = 1$ has everywhere positive Gauss curvature. This definition of regularity is equivalent to the condition that the Euler equations of the corresponding local non-parametric variational problems are elliptic. Note that conditions (i) and (ii) imply that Σ is a closed surface, star-shaped about the origin, since if $\mathscr{J}(\boldsymbol{p}) = 1$, then $\boldsymbol{p} = \boldsymbol{N}/\mathscr{J}(\boldsymbol{N})$, where $\boldsymbol{N}$ is a unit vector in the direction of $\boldsymbol{p}$. Our results apply to solutions of the Euler equations of (1), hence we shall always use the term "solution", or "solution surface", in that sense. If $\boldsymbol{N}(q)$ is the unit normal at the point q of a solution surface S, $\boldsymbol{p}(q) = \boldsymbol{N}(q)/\mathscr{J}(\boldsymbol{N}(q))$ defines a mapping of S into Σ which is a generalization of the classical spherical image mapping. We shall show that the behavior of S in a neighborhood of a point q is governed by the second fundamental form of Σ at the image point $\boldsymbol{p}(q)$.

It is our purpose here to study the geometrical relationships between the surface Σ and a solution surface. We shall show that the mapping $\boldsymbol{p}(q)$ determines a non-trivial linear mapping of the differential forms on Σ into those on a solution surface S; and that in the case of regular variational problems this mapping in turn gives rise to a canonical mapping of S onto the solution of a non-parametric, uniformly elliptic variational problem.

The principal application is to show that solutions of variational problems which satisfy conditions (i), (ii), (iii) behave qualitatively like minimal surfaces, in the same sense that solutions of uniformly elliptic equations behave like solutions of the Laplace equation. We shall show that the mapping $\boldsymbol{p}(q)$ of a solution surface S into Σ possesses a property which is a natural generalization

of the conformality of the spherical mapping of a minimal surface into the unit sphere (Theorem 1). It is an immediate consequence of our methods that the quasi-linear equations satisfied by solutions of the form $z=\varphi(x, y)$ are "equations of minimal surface type" in the sense of FINN [1] and that they can be written in the form $(A(u, v))_x+(B(u, v))_y=0$, where $u=\varphi_x$, $v=\varphi_y$, and A and B are bounded functions*. Theorems of FINN [1], [2], [3] may therefore be applied to show that such solutions share with non-parametric minimal surfaces the properties (a) the gradient of φ at a point (x_0, y_0) can be estimated in terms of the distance r to the boundary and a bound M on the quantity $|\varphi(x, y)-\varphi(x_0, y_0)|$**; (b) there are no solutions with isolated singularities; and (c) the boundary value problem can be solved for continuous data on convex curves. Again in the case of solutions of the form $z=\varphi(x, y)$, a theorem of the author [4], can be applied to show that the gradient of a solution defined in the exterior of a circle goes Hölder continuously to a finite limit at infinity. In the minimal surface case, the latter property and property (b) were established by BERS [5], and (a) and (c) were given by FINN.

Let S be a simply connected minimal surface whose normal direction at each point makes an angle $\geqq\beta$ with a fixed direction. Let q be a point of S such that the distance along S to the boundary is $\geqq d$. Let α be the angle between the normal at q and the fixed direction. OSSERMAN [7] has recently shown that the Gauss curvature $K(q)$ of S at q then satisfies an inequality

$$|K(q)| \leqq \frac{1}{d^2}\left[\frac{4(r^2-a^2)\,h(a, r)}{r(1-a^2)^2}\right]^2, \tag{2}$$

where $r=\cot\beta/2$, $a=\cot\alpha/2$, and $h(a, r)$ is a known function. We shall show that the Gauss curvature of a solution of a regular parametric variational problem of the type (1) satisfies an inequality

$$|K(q)| \leqq C\,\frac{(\alpha-\beta)^2}{d^2}, \tag{3}$$

where α, β, q, d are as defined above, and C is a constant which depends only on β and $\mathscr{J}(\boldsymbol{p})$. This inequality is qualitatively similar to OSSERMAN's inequality and may be interpreted as implying that the curvature at a point q of a solution surface S is small not only when the distance to the boundary is large, but also when the spherical image of the point q is close to the boundary of the spherical image of S. In the case of solutions of the form $z=\varphi(x, y)$, the inequality (3) may be put in the form

$$|K(x_0, y_0)| \leqq \frac{C_1}{W_0^2\, r^2}, \tag{4}$$

where r is the radius of the largest circle about the point (x_0, y_0) in which φ is defined, and $W_0=\left(1+\varphi_x^2(x_0, y_0)+\varphi_y^2(x_0, y_0)\right)^{\frac{1}{2}}$. A characteristic property of solutions of equations of minimal surface type is that the spherical mapping is quasi-conformal, and this in turn implies that the ratio of the magnitudes of the

* If an equation can be written in the form $(A(u, v))_x+(B(u, v))_y=0$, then the pair (A, B) is said to be a conservation law of the equation (see LOEWNER [6]).

** It will be shown in another paper that an estimate of the gradient in terms of r and M can be obtained from the inequality (4), below, by elementary means.

principal curvatures is bounded above and below by positive constants. Combining the latter property with (4) results in

$$r_0^2 + 2s_0^2 + t_0^2 \leqq \frac{C_2 W_0^4}{r^2}, \tag{5}$$

where $r_0 = \varphi_{xx}(x_0, y_0)$, $s_0 = \varphi_{xy}(x_0, y_0)$, $t_0 = \varphi_{yy}(x_0, y_0)$, and W_0 and r are as defined above. It is an immediate consequence of the inequalities (3), (4) and (5) that solutions of the form $z = \varphi(x, y)$, defined for all x, y, are necessarily of the form $z = ax + by + c$; and more generally that connected complete solution surfaces whose normal directions omit a neighborhood of a fixed direction are necessarily planes. In the minimal surface case, the former result was first proven by BERNSTEIN [*8*], and the latter was conjectured by NIRENBERG and proven by OSSERMAN [*9*]. For non-parametric minimal surfaces, estimates of the second derivatives and of the curvature were originally given by HEINZ [*10*], and the inequality (4) was given by HOPF [*11*]. An elementary proof of the HOPF inequality was given by NITSCHE [*12*]. A generalization of the HEINZ estimates to a class of non-parametric variational problems has been given by the author [*4*].

Our methods permit the formulation of a problem in partial differential equations which is a generalization of the parametric variational problem (1) in the same sense that the quasi-linear equation $a(\varphi_x, \varphi_y)\,\varphi_{xx} + 2b(\varphi_x, \varphi_y)\,\varphi_{xy} + c(\varphi_x, \varphi_y)\,\varphi_{yy} = 0$ is a generalization of the Euler equation of the non-parametric variational problem $\delta \int F(\varphi_x, \varphi_y)\, dx\, dy = 0$.

If $(A(u, v), B(u, v))$ is a conservation law of a non-parametric variational problem $\delta \int F(u, v)\, dx\, dy = 0$, then $B\,dx - A\,dy$ is an exact differential on each solution surface $z = \varphi(x, y)$. LOEWNER [*6*] has shown that the conservation laws of a non-parametric variational problem of this type are the solutions of a certain first-order linear system. Now let S be an arbitrary solution surface of a regular parametric variational problem of the type of (1). In some neighborhood of each point q of S, S can be represented in the form $z = \varphi(x, y)$ for a suitable coordinate system x, y, z; and it follows that the function $\varphi(x, y)$ is a solution of a non-parametric variational problem of the above type. The system of LOEWNER for the conservation laws of this problem is a (non-uniformly) elliptic system and hence in the neighborhood of a given point of the u, v plane has infinitely many solutions. Thus in a neighborhood of the point q of S there exist many exact differentials which arise from conservation laws of the local non-parametric problem. We shall investigate the global existence of such exact differentials, and we shall show that they arise globally from certain 1-forms on the surface Σ. These 1-forms are a generalization to the parametric case, of the conservation laws of the non-parametric problems. In the case of regular parametric variational problems, we shall characterize the generalized conservation laws explicitly in terms of solutions of a single uniformly elliptic system. In the minimal surface case it will be shown that (a) the conservation laws giving rise to exact differentials on minimal surfaces whose spherical images omit a neighborhood are in $1-1$ correspondence with the class of analytic functions regular in a disc; (b) the conservation laws giving rise to exact differentials on minimal surfaces whose spherical images omit a single point are in $1-1$ correspondence with the class of entire analytic functions; and (c) the conservation laws giving rise to exact

differentials on minimal surfaces whose spherical images cover the sphere are in $1-1$ correspondence with the set of complex polynomials of degree $\leqq 2$.

II. Assume that $\mathcal{J}(\boldsymbol{p})$ satisfies conditions (i) and (ii); then as indicated above, the surface Σ defined by $\mathcal{J}(\boldsymbol{p})=1$ is a closed surface, star-shaped about the origin, and the distance $R(\boldsymbol{N})$ from the origin to the point of Σ in the direction of the unit vector N is given by $R(\boldsymbol{N})=1/\mathcal{J}(\boldsymbol{N})$. Let S be an orientable surface of class C^2 in Euclidean 3-space, and assume that one of the normal directions to S has been chosen to be the positive normal direction. Let q be an arbitrary point of S, and let $\boldsymbol{N}(q)$ be the unit normal vector to S at q. The mapping $\boldsymbol{p}(q)=R(\boldsymbol{N})\boldsymbol{N}$ is then a mapping of S into Σ which is a generalization of the spherical mapping into the unit sphere. This mapping will be called the "normal mapping". Associated with a differentiable mapping of a manifold A into a manifold B, there is a trivial linear mapping of the differential forms on B into those on A; *i.e.*, if $x_1, \ldots, x_n, y_1, \ldots, y_m$ are the local coordinates of corresponding points of B and A respectively, then dx_i is mapped to $\Sigma(dx_i/dy_j)\,dy_j$ under the trivial mapping. We now show that the normal mapping of S into Σ determines a non-trivial linear mapping Φ of the differential forms on Σ into those on S. Assume that S and Σ lie in different spaces X and Y respectively, but that directions in X and Y have been identified so that the normal mapping is well defined. In order to define Φ, we define specific systems of coordinate patches and local coordinates on S and Σ. We take for the coordinate patches of Σ the open regions which lie on one side of planes through the origin. For the coordinate patches of S, we take open regions of S whose images under the normal mapping lie in coordinate patches of Σ. A coordinate patch V of S whose image under the normal mapping lies in a coordinate patch U of Σ will be said to "correspond" to U. Thus, let σ, τ, ω be a coordinate system in Y, and let x, y, z be a coordinate system for X, with the x, y, z axes parallel respectively to the σ, τ, ω axes. Let U be the coordinate patch of Σ for which $\omega<0$. A coordinate patch V of S which corresponds to U is thus a region of S in which the normals make an acute angle with the negative z-axis. In V, therefore, S can be represented in the form $z=\varphi(x, y)$. We take x, y to be the local coordinates in V; and in U we take as local coordinates u, v, where

$$u=-\frac{\sigma}{\omega}, \qquad v=-\frac{\tau}{\omega}. \tag{6}$$

Equations (6) define a $1-1$ mapping of U onto the u, v plane; since if σ, τ, ω are the coordinates of a point P of U, u, v are the σ, τ coordinates of the point obtained by projecting P through the origin into the plane $\omega=-1$. Moreover, the Jacobians $\partial(\tau, \omega)/\partial(u, v)$, $\partial(\omega, \sigma)/\partial(u, v)$, $\partial(\sigma, \tau)/\partial(u, v)$ do not simultaneously vanish, since conditions (i), (ii) imply that the outer normal to Σ always makes an acute angle with the radial direction; *i.e.*, $\mathcal{J}=\sigma\mathcal{J}_\sigma+\tau\mathcal{J}_\tau+\omega\mathcal{J}_\omega>0$. Putting $F(u, v)=-1/\omega$ and $u=\varphi_x$, $v=\varphi_y$, we see that in V the normal mapping has the form

$$\sigma=u/F, \qquad \tau=v/F, \qquad \omega-1/F. \tag{7}$$

Let $\bar\sigma, \bar\tau, \bar\omega$ be a system of coordinates for Y, obtained from the σ, τ, ω coordinate system by a rotation defined by a rotation matrix $A=(a_{ij})$. Let $\bar U$ be the coordi-

nate patch defined by $\overline{\omega}<0$, and assume that $U \cap \overline{U} \neq 0$. Noting that the inverse of A equals the transpose of A, the relations between u, v and the local coordinates $\bar{u}, \bar{v}$ in $\overline{U}$ are easily seen to be

$$\text{(8)} \qquad \begin{aligned} \bar{u} &= \frac{a_{11}u + a_{12}v - a_{13}}{\Delta}, & \bar{v} &= \frac{a_{21}u + a_{22}v - a_{23}}{\Delta} \\ u &= \frac{a_{11}\bar{u} + a_{21}\bar{v} - a_{31}}{\overline{\Delta}}, & v &= \frac{a_{12}\bar{u} + a_{22}\bar{v} - a_{32}}{\overline{\Delta}} \end{aligned}$$

where $\Delta = -a_{31}u - a_{32}v + a_{33}$, $\overline{\Delta} = -a_{13}\bar{u} - a_{23}\bar{v} + a_{33}$. The region of the u, v plane corresponding to $U \cap \overline{U}$ is the half plane $\Delta > 0$, since in $U \cap U$, $\omega, \overline{\omega} < 0$ and $\overline{\omega} = \Delta\omega$. A calculation shows also that $\Delta = 1/\overline{\Delta}$, and that $\partial(u, v)/\partial(\bar{u}, \bar{v}) = \Delta^3$. Let $\overline{V}$ be a coordinate patch of S which corresponds to $\overline{U}$ and intersects V. Let $\bar{x}, \bar{y}, \bar{z}$ be the coordinate system for X obtained from the coordinate system x, y, z by the rotation with matrix A. $\bar{x}, \bar{y}$ are then local coordinates in $\overline{V}$, and in $\overline{V}$ S can be represented in the form $\bar{z} = \overline{\varphi}(\bar{x}, \bar{y})$. The relations between $x, y, \varphi(x, y)$ and $\bar{x}, \bar{y}, \overline{\varphi}(\bar{x}, \bar{y})$ in $V \cap \overline{V}$ are then given by

$$\text{(9)} \qquad \begin{aligned} x &= a_{11}\bar{x} + a_{21}\bar{y} + a_{31}\overline{\varphi}(\bar{x}, \bar{y}) \\ y &= a_{12}\bar{x} + a_{22}\bar{y} + a_{32}\overline{\varphi}(\bar{x}, \bar{y}) \\ \varphi(x, y) &= a_{13}\bar{x} + a_{23}\bar{y} + a_{33}\overline{\varphi}(\bar{x}, \bar{y}). \end{aligned}$$

We now put $u = \varphi_x$, $v = \varphi_y$, and calculate $\bar{u}, \bar{v}$ according to (8). It then follows from (9) that $\bar{u} = \overline{\varphi}_{\bar{x}}$, $\bar{v} = \overline{\varphi}_{\bar{y}}$. A calculation now shows that in $V \cap \overline{V}$,

$$\text{(10)} \qquad \begin{aligned} \Delta^2 x_{\bar{x}} &= v_{\bar{v}}, & \Delta^2 y_{\bar{x}} &= -u_{\bar{v}} \\ \Delta^2 x_{\bar{y}} &= -v_{\bar{u}}, & \Delta^2 y_{\bar{y}} &= u_{\bar{u}}. \end{aligned}$$

We now define the Φ-mapping for 1-forms. Let h be a 1-form on Σ. In U, h is then given by an expression $h = \vartheta(u, v)\,du + \Lambda(u, v)\,dv$; and in $\overline{U}$, by an expression $h = \overline{\vartheta}(\bar{u}, \bar{v})\,d\bar{u} + \overline{\Lambda}(\bar{u}, \bar{v})\,d\bar{v}$. In $U \cap \overline{U}$, ϑ, Λ and $\overline{\vartheta}, \overline{\Lambda}$ must satisfy

$$\text{(11)} \qquad \overline{\vartheta} = \vartheta u_{\bar{u}} + \Lambda v_{\bar{u}}, \quad \overline{\Lambda} = \vartheta u_{\bar{v}} + \Lambda v_{\bar{v}}.$$

Φh is now defined in any coordinate patch V, corresponding to U, by

$$\text{(12)} \qquad \Phi h = F^2[\Lambda(u(x, y), v(x, y))\,dx - \vartheta(u(x, y), v(x, y))\,dy].$$

This definition is consistent since (8) implies that $\overline{\omega}/\omega = \Delta = F/\overline{F}$, and (10) and (11) imply that in $V \cap \overline{V}$

$$\text{(13)} \qquad \begin{aligned} \Phi h = F^2(\Lambda\,dx - \vartheta\,dy) &= F^2[(\Lambda x_{\bar{x}} - \vartheta y_{\bar{y}})\,d\bar{x} + (\Lambda x_{\bar{y}} - \vartheta y_{\bar{y}})\,d\bar{y}] \\ &= \overline{F}^2(\overline{\Lambda}\,d\bar{x} - \overline{\vartheta}\,d\bar{y}). \end{aligned}$$

Similarly, if f is a quadratic differential form given in U by

$$\text{(14)} \qquad f = A\,du^2 + 2B\,du\,dv + C\,dv^2,$$

Φf is defined in each coordinate patch V of S by

$$\text{(15)} \qquad \Phi f = F^4(C\,dx^2 - 2B\,dx\,dy + A\,dy^2).$$

The consistency of this definition follows as in the case of 1-forms. The Φ-mapping is obviously linear and may be defined analogously for differential forms of higher order.

If f is a differential form on Σ, we shall denote f considered as a differential form on S (*i.e.*, the image of f under the trivial mapping) by the same symbol. The following theorem now gives the connection between the Φ-mapping and the variational problem (1).

Theorem 1. *Let f denote the second fundamental form of Σ; then a surface S of class C^2 is a solution of the Euler equations of the variational problem (1) if and only if, at every point q of S,*

$$f/\Phi f = -R^4(\boldsymbol{N}(q))\, K(q), \tag{16}$$

where $K(q)$ is the Gauss curvature of S at q.

Proof. Let V be a coordinate patch of S corresponding to a coordinate patch U of Σ. Assume that S is given in parametric form with parameters s, t and that a region V^* of the s, t plane corresponds to the coordinate patch V of S. Let $dA(q)$ denote the element of area on S at the point q. Then

$$\iint_{V^*} \mathcal{F}(\boldsymbol{p})\, ds\, dt = \int_V \left[1/R(\boldsymbol{N}(q))\right] dA(q) = \iint_{V'} F(u, v)\, dx\, dy \tag{17}$$

where V' denotes the projection of V onto the plane of its local coordinates x, y. It now follows that S is a solution in V of the Euler equations of (1) if and only if the function $\varphi(x, y)$ which represents S in V is a solution of the Euler equation of the non-parametric variational problem $\delta \iint F(u, v)\, dx\, dy = 0$; *i.e.*, if and only if

$$F_{uu}\varphi_{xx} + 2F_{uv}\varphi_{xy} + F_{vv}\varphi_{yy} = 0. \tag{18}$$

Let $L(\varphi)$ denote the expression $F_{uu}\varphi_{xx} + 2F_{uv}\varphi_{xy} + F_{vv}\varphi_{yy}$. A calculation then shows that

$$\begin{aligned} F_{uu}\, du^2 + 2F_{uv}\, du\, dv + F_{vv}\, dv^2 = L(\varphi)\,(\varphi_{xx}\, dx^2 + 2\varphi_{xy}\, dx\, dy + \varphi_{yy}\, dy^2) + \\ + (\varphi_{xy}^2 - \varphi_{xx}\varphi_{yy})\,(F_{vv}\, dx^2 - 2F_{uv}\, dx\, dy + F_{uu}\, dy^2). \end{aligned} \tag{19}$$

It will be a consequence of Lemma 1, proved below, that in U the second fundamental form f is given by

$$f = \lambda(u, v)\,(F_{uu}\, du^2 + 2F_{uv}\, du\, dv + F_{vv}\, dv^2) \tag{20}$$

where $\lambda(u, v)$ is a positive function. (Alternatively, equation (20) may be obtained by a calculation showing in addition that $\lambda = (1/F)\,(F_u^2 + F_v^2 + (F - uF_u - vF_v)^2)^{-\frac{1}{2}}$.) Since $W/F = R$, where $W = (1 + u^2 + v^2)^{\frac{1}{2}}$, the result now follows from equations (15), (19) and (20). In the minimal surface case, (16) reduces to the well known formula $K(q) = -ds_0^2/ds_1^2$; since Σ is then the unit sphere, $f = ds_0^2$ is the Euclidean metric on Σ, and $\Phi f = ds_1^2$ is the Euclidean metric on S.

If $\mathcal{F}(\boldsymbol{p})$ is regular, the second fundamental form f is positive definite and clearly so is Φf. We may therefore take f, Φf to be Riemannian metrics on Σ and S respectively. Theorem 1 now implies that the normal mapping of S into Σ is conformal in the metrics Φf, f.

Lemma 1. *Let Σ be a surface of class C^2, represented parametrically in the form $\mathfrak{r}(u, v) = (\sigma(u, v),\ \tau(u, v),\ \omega(u, v))$. Let $\mathfrak{r}^* = P\mathfrak{r}$ be a projective transformation of the space. $\mathfrak{r}^*(u, v) = P\mathfrak{r}(u, v)$ is then a parametric representation of the surface Σ^* which is the image of Σ under P. The conclusion is that the second fundamental forms of Σ and Σ^* are proportional at corresponding points.*

Proof. The most general projective transformation of σ, τ, ω space is given by

$$
\begin{aligned}
\sigma^* &= (1/\delta)\,(P_{11}\sigma + P_{12}\tau + P_{13}\omega + P_{14}),\\
\tau^* &= (1/\delta)\,(P_{21}\sigma + P_{21}\tau + P_{23}\omega + P_{24}),\\
\omega^* &= (1/\delta)\,(P_{31}\sigma + P_{32}\tau + P_{33}\omega + P_{34}),
\end{aligned}
\tag{21}
$$

where $\delta = P_{41}\sigma + P_{42}\tau + P_{43}\omega + P_{44}$, and $P = (P_{ij})$ is a 4×4 matrix with $\det(P_{ij}) = 1$. Note that we are using the symbol P ambiguously to denote both the projective transformation and the 4×4 matrix. The precise meaning will be clear from the context. Let $\mathfrak{z}$, $\mathfrak{z}^*$ denote the 4-vectors $(\sigma, \tau, \omega, 1)$, $(\sigma^*, \tau^*, \omega^*, 1)$ respectively. It then follows that $\mathfrak{z}^* = (1/\delta)\,P\mathfrak{z}$. Let $L\,du^2 + 2M\,du\,dv + N\,dv^2$, $L^*du^2 + 2M^*du\,dv + N^*dv^2$ denote the second fundamental forms of Σ, Σ^* respectively, and let $\begin{vmatrix} \boldsymbol{A} \\ \boldsymbol{B} \\ \boldsymbol{C} \\ \boldsymbol{D} \end{vmatrix}$ denote the 4×4 matrix whose rows are the 4-vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$. Then

$$
\begin{aligned}
\mathfrak{r}^*_{uu} \cdot \mathfrak{r}^*_u \times \mathfrak{r}^*_v &= \det \begin{vmatrix} \mathfrak{z}^*_{uu} \\ \mathfrak{z}^*_u \\ \mathfrak{z}^*_v \\ \mathfrak{z}^* \end{vmatrix}\\
&= \det \begin{vmatrix} (1/\delta)\,P\mathfrak{z}_{uu} + 2(1/\delta)_u\,P\mathfrak{z}_u + (1/\delta)_{uu}\,P\mathfrak{z} \\ (1/\delta)\,P\mathfrak{z}_u + (1/\delta)_u\,P\mathfrak{z} \\ (1/\delta)\,P\mathfrak{z}_v + (1/\delta)_v\,P\mathfrak{z} \\ (1/\delta)\,P\mathfrak{z} \end{vmatrix}\\
&= (1/\delta)^4 \det \begin{vmatrix} P\mathfrak{z}_{uu} \\ P\mathfrak{z}_u \\ P\mathfrak{z}_v \\ P\mathfrak{z} \end{vmatrix} = (1/\delta)^4 \det(P_{ij}) \det \begin{vmatrix} \mathfrak{z}_{uu} \\ \mathfrak{z}_u \\ \mathfrak{z}_v \\ \mathfrak{z} \end{vmatrix}\\
&= (1/\delta)^4\,(\mathfrak{r}_{uu} \cdot \mathfrak{r}_u \times \mathfrak{r}_v).
\end{aligned}
\tag{22}
$$

Similar equations hold for $\mathfrak{r}^*_{uv} \cdot \mathfrak{r}^*_u\,\mathfrak{r}^*_v$ and $\mathfrak{r}^*_{vv} \cdot \mathfrak{r}^*_u\,\mathfrak{r}^*_v$; thus $L^*/L = M^*/M = N^*/N = (1/\delta)^4 (|\mathfrak{r}_u \times \mathfrak{r}_v| / |\mathfrak{r}^*_u \times \mathfrak{r}^*_v|) = \chi(u, v)$. The step needed in the proof of Theorem 1 follows since the surface $w = F(u, v)$ is the image under a projective transformation of the neighborhood U of Σ. This lemma is more elaborate than is necessary for the proof of Theorem 1; however, we shall need it again later.

III. The parametric variational problems considered above may be generalized to problems in partial differential equations. Let Σ_1 be the unit sphere, and let f be a quadratic differential form on Σ. Let S be a surface of class C^2,

and define coordinate patches and local coordinates on S and Σ_1 as before. We shall say that S is a solution of the problem associated with f if at each point of S

$$f/\Phi f = -K \tag{23}$$

where K is the Gauss curvature of S. Thus let U be a coordinate patch of Σ_1, and let V be a coordinate patch of S which corresponds to U. It may be readily verified in this case that $F(u, v)$ has the same form in every coordinate patch and is given by

$$F(u, v) = -1/\omega = (1 + u^2 + v^2)^{\frac{1}{2}} = W. \tag{24}$$

In U, let f be given by

$$f = a(u, v)\, du^2 + 2b(u, v)\, du\, dv + c(u, v)\, dv^2. \tag{25}$$

Φf is then given in V by

$$\Phi f = W^4(c\, dx^2 - 2b\, dx\, dy + a\, dy^2). \tag{26}$$

A calculation similar to (19) now shows that the part of S lying in V is a solution if and only if the function $\varphi(x, y)$ representing S in V satisfies the quasi-linear equation

$$a\, \varphi_{xx} + 2b\, \varphi_{xy} + c\, \varphi_{yy} = 0. \tag{27}$$

If f is a positive definite quadratic form, then Φf is positive definite also, and f, Φf may be taken to be Riemannian metrics on Σ_1, S respectively. Again, if S is a solution, the spherical mapping is conformal in the metrics Φf, f. The following problem suggests itself. Given a Riemannian metric ds_0^2 on the unit sphere, and a closed Jordan curve in space; does there exist a surface S which spans the curve such that $ds_0^2/\Phi\, ds_0^2 = -K$?

IV. If two Riemannian metrics are defined on the same two-dimensional manifold, the infinitesimal circle at a given point in one metric is an infinitesimal ellipse in the other metric. If the ratio δ of the major axis to the minor axis is uniformly bounded for all points of the manifold, the metrics are said to be quasi-conformally related. If x_1, x_2 are local variables in a neighborhood of the manifold and the metrics are given by $a_1\, dx_1^2 + 2b_1\, dx_1\, dx_2 + c_1\, dx_2^2$, $a_2\, dx_1 + 2b_2\, dx_1\, dx_2 + c_2\, dx_2^2$, then a formal calculation shows that

$$\frac{1}{2}(\delta + 1/\delta) = E = \frac{1}{2}\, \frac{a_1 c_2 + a_2 c_1 - 2b_1 b_2}{(a_1 c_1 - b_1^2)^{\frac{1}{2}}\, (a_2 c_2 - b_1^2)^{\frac{1}{2}}}. \tag{28}$$

The quantitiy E is said to be the eccentricity of one metric with respect to the other, and clearly the metrics are quasi-conformally related if and only if $\operatorname{Sup} E < \infty$.* If the manifold is compact and the metrics are continuous, then clearly the metrics are necessarily quasi-conformally related. In particular any continuous metric ds_0^2 on the unit sphere is quasi-conformally related to the Euclidean metric. In terms of the local coordinates u, v in a coordinate patch U of the unit sphere Σ_1, the Euclidean metric is given by $ds^2 = (1/W^4)\,[(1+v^2)\, du^2 - 2uv\, du\, dv + (1+u^2)\, dv^2]$. It follows therefore that if f is a continuous, positive definite quadratic form on Σ_1 which is given in U by $f = a\, du^2 + 2b\, du\, dv + c\, dv^2$,

* The criterion (28) is due to Finn. For a geometrical interpretation see [1].

then there is a constant E_0 which is independent of the particular coordinate patch U such that

(29)
$$E = \frac{1}{2} \frac{a(1+u^2)+c(1+v^2)+2uvb}{(ac-b^2)^{\frac{1}{2}}(1+u^2+v^2)^{\frac{1}{2}}} \leqq E_0.$$

This condition on the functions a, b, c is precisely the condition that equation (27) be an equation of minimal surface type.

Now let Σ be the surface $\mathscr{J}(\boldsymbol{p}) = 1$, where $\mathscr{J}(\boldsymbol{p})$ satisfies conditions (i) and (ii). If we map Σ onto the unit sphere Σ_1 by projection through the origin, each coordinate patch U of Σ is mapped onto a coordinate patch U' of Σ_1, and corresponding points of U and U' have the same local coordinates u, v. We may transplant the second fundamental form of Σ to Σ_1 by means of the trivial mapping, and in terms of the local coordinates u, v it will be given by the same expression whether regarded as a quadratic form on Σ or as a quadratic form on Σ_1. Thus the variational problem $\delta \int \mathscr{J}(\boldsymbol{p})\, ds\, dt = 0$ may be taken to be a special case of the problems discussed in Section III. If $\mathscr{J}(\boldsymbol{p})$ is regular, then the second fundamental form of Σ is positive definite and, considered as a metric on Σ_1, is quasi-conformally related to the Euclidean metric. Equation (27) coincides in this case with equation (18) and, by the above remarks, is an equation of minimal surface type. According to the results of Section II, equation (18) is the Euler equation of a non-parametric variational problem $\delta \int F(u, v)\, dx\, dy = 0$, where the surface $w = F(u, v)$ is obtained from the neighborhood U of the surface Σ by the projective transformation $u = -\sigma/\omega$, $v = -\tau/\omega$, $F(u, v) = -1/\omega$. Equation (18) may therefore be written in the form $(F_u)_x + (F_v)_y = 0$, and therefore the pair (F_u, F_v) is a conservation law of equation (18).

We now show that if $\mathscr{J}(\boldsymbol{p})$ is regular, then equation (18) satisfies the hypothesis of the theorems of Finn and of the theorem of the author which are mentioned in the Introduction. The theorems of Finn apply to equations of the type of equation (27) which are of minimal surface type and have a bounded conservation law. The theorem of the author on the other hand applies to the Euler equations of non-parametric variational problems of the above type, which are of minimal surface type, and which satisfy the condition that the quantity $|F - uF_u - vF_v|$ is bounded. That equation (18) has these additional properties may be seen immediately from the following observation. If p is a point of U with local coordinates u, v, then the distance $r(p)$ from the origin to the tangent plane to Σ at p is given by

(30)
$$r(p) = 1/\left(F_u^2 + F_v^2 + (F - uF_u - vF_v)^2\right)^{\frac{1}{2}}.$$

If $\mathscr{J}(\boldsymbol{p})$ is regular, then Σ is a closed convex surface containing the origin, and $r(p)$ is clearly bounded above and below by positive constants. The result now follows from (30). We have now proved the following theorem.

Theorem 2. *Let S be a solution surface of the form $z = \varphi(x, y)$ of the parametric variational problem $\delta \int \mathscr{J}(\boldsymbol{p})\, ds\, dt = 0$. Assume that $\mathscr{J}(\boldsymbol{p})$ is of class C^2 for all $\boldsymbol{p} \neq 0$, and that $\mathscr{J}(\boldsymbol{p})$ is positive homogeneous of degree 1, positive definite, and regular. The Euler equation of the non-parametric variational problem $\delta \int F(u,v)\, dx\, dy$ satisfied by $\varphi(x, y)$ is then an equation of minimal surface type, having a bounded conservation law and satisfying the condition that $|F - uF_u - vF_v|$ is bounded.*

V. We return now to the study of parametric variational problems satisfying conditions (i) and (ii). Σ is a closed star-shaped surface of class $C^{2-\alpha}$ and f is the second fundamental form of Σ. Let S be a solution surface, and let h be a 1-form on Σ. It is natural to ask if there are conditions on h which will imply that Φh is an exact differential on S. Again let U be a coordinate patch on Σ with local variables u, v. Let V be a coordinate patch on S with local variables x, y, which corresponds to U. Let h be given in U by

$$h = \vartheta(u, v)\, du + \Lambda(u, v)\, dv. \tag{31}$$

Φh is then given in V by

$$\Phi h = F^2(\Lambda\, dx - \vartheta\, dy). \tag{32}$$

Thus Φh is exact in V if $(F^2\vartheta)_x + (F^2\Lambda)_y = 0$; *i.e.*, if $F^2\vartheta$, $F^2\Lambda$ are the components of a conservation law of the quasi-linear equation (18) which is satisfied by the function $\varphi(x, y)$ representing S in V. According to the theory of LOEWNER [6], this will be the case if $F^2\vartheta$, $F^2\Lambda$ satisfy the linear system

$$\begin{aligned} F_{vv}(F^2\vartheta)_u &= F_{uu}(F^2\Lambda)_v, \\ 2F_{uv}(F^2\vartheta)_u &= F_{uu}[(F^2\vartheta)_v + (F^2\Lambda)_u]. \end{aligned} \tag{33}$$

If this condition holds in all coordinate patches of Σ, then Φh is an exact differential on all of S. In the following lemma we show that with reference to a fixed coordinate system in X and Y, the images under the Φ-mapping of the differentials of the coordinate functions on Σ are exact differentials on S and that the differentials of the coordinate functions on S are the images under the Φ-mapping of differentials on Σ.

Lemma 2. *Let σ, τ, ω; x, y, z be coordinate systems for Y and X respectively with the σ, τ, ω axes parallel respectively to the x, y, z axes. It then follows that* (a) *$d\xi = \Phi(d\tau)$, $d\eta = \Phi(-d\sigma)$, $d\psi = \Phi(d\omega)$ are exact differentials on S and that* (b) *$dx = \Phi(\tau\, d\omega - \omega\, d\tau)$, $dy = \Phi(\omega\, d\sigma - \sigma\, d\omega)$, and $dz = \Phi(\sigma\, d\tau - \tau\, d\sigma)$.*

Proof of part (a). Let $\overline{V}$ be an arbitrary coordinate patch of S, and let $\overline{U}$ be a coordinate patch of Σ to which $\overline{V}$ corresponds. Let $\bar\sigma, \bar\tau, \bar\omega$ be a coordinate system for Y such that $\overline{U}$ is defined by $\bar\omega < 0$, and let $\bar x, \bar y, \bar z$ be a parallel coordinate system for X. $\bar x, \bar y$ are then the local coordinates in $\overline{V}$. $d\sigma, d\tau, d\omega$ are linear combinations of $d\bar\sigma, d\bar\tau, d\bar\omega$; hence it suffices to show that $\Phi(d\bar\sigma)$, $\Phi(d\bar\tau)$, $\Phi(d\bar\omega)$ are exact in $\overline{V}$. In terms of the local coordinates $\bar u, \bar v$ in $\overline{U}$, however,

$$\begin{aligned} d\bar\sigma &= (1/\overline{F})^2[(\overline{F} - \bar u\overline{F}_{\bar u})\, d\bar u - \bar u\overline{F}_{\bar v}\, d\bar v] \\ d\bar\tau &= (1/\overline{F})^2[-\bar v\overline{F}_{\bar u}\, d\bar u + (\overline{F} - \bar v\overline{F}_{\bar v})\, d\bar v] \\ d\bar\omega &= (1/\overline{F})^2[\overline{F}_{\bar u}\, d\bar u + \overline{F}_{\bar v}\, d\bar v], \end{aligned} \tag{34}$$

hence

$$\begin{aligned} \Phi(-d\bar\sigma) &= \bar u\overline{F}_{\bar v}\, d\bar x + (\overline{F} - \bar u\overline{F}_{\bar u})\, d\bar y \\ \Phi(d\bar\tau) &= (\overline{F} - \bar v\overline{F}_{\bar v})\, d\bar x + \bar v\overline{F}_{\bar u}\, d\bar y \\ \Phi(d\bar\omega) &= \overline{F}_{\bar v}\, d\bar x - \overline{F}_{\bar u}\, d\bar y. \end{aligned} \tag{35}$$

This proves part (a), since it is well known that $-\bar{u}\bar{F}_{\bar{v}}, \bar{F}-\bar{u}\bar{F}_{\bar{u}}; \bar{F}-\bar{v}\bar{F}_{\bar{v}}, -\bar{v}\bar{F}_{\bar{u}}$, and $\bar{F}_{\bar{u}}, \bar{F}_{\bar{v}}$ are the components of conservation laws of the equation

$$\bar{F}_{\bar{u}\bar{u}}\,\bar{\varphi}_{\bar{x}\bar{x}} + 2\bar{F}_{\bar{u}\bar{v}}\,\bar{\varphi}_{\bar{x}\bar{y}} + \bar{F}_{\bar{v}\bar{v}}\,\bar{\varphi}_{\bar{y}\bar{y}} = 0. \tag{36}$$

We remark that the exactness of $d\xi$, $d\eta$, $d\psi$ can be inferred from the parametric form of the Euler equations and hence is well known. Our method has the advantage that it yields quantitative information about the functions ξ, η, ψ which arise by integration.

Proof of part (b). Let $\bar{V}$, $\bar{U}$; $\bar{x}, \bar{y}, \bar{z}$; $\bar{\sigma}, \bar{\tau}, \bar{\omega}$; $\bar{u}, \bar{v}$, be defined as in the proof of part (a). Note that the assertion of part (b) may be expressed in vector notation as $d\boldsymbol{r} = \Phi(\boldsymbol{R} \times d\boldsymbol{R})$, where $\boldsymbol{r}, \boldsymbol{R}$ denote the position vectors of points of S and Σ respectively. In $\bar{V}$ it may be readily verified by calculation that $d\bar{x} = \Phi(\bar{\tau}\, d\bar{\omega} - \bar{\omega}\, d\bar{\tau})$, $d\bar{y} = \Phi(\bar{\omega}\, d\bar{\sigma} - \bar{\sigma}\, d\bar{\omega})$, $d\bar{z} = \Phi(\bar{\sigma}\, d\bar{\tau} - \bar{\tau}\, d\bar{\sigma})$. The result now follows from the geometrical invariance of $d\boldsymbol{r}$ and $\boldsymbol{R} \times d\boldsymbol{R}$ under rotation.

In Lemmas 3 and 4, we continue to assume that $d\xi$, $d\eta$, and $d\psi$ are obtained via the Φ-mapping from the differentials of the coordinate functions on Σ with respect to a fixed coordinate system $\bar{\sigma}, \bar{\tau}, \bar{\omega}$ for the space Y. We shall assume that S is a simply connected solution surface so that the differentials $d\xi$, $d\eta$, $d\psi$ give rise, by integration, to single-valued functions ξ, η, ψ on S. We now derive some properties of the mapping of S into the plane which is defined by the functions ξ, η.

Lemma 3. *Let q be a point of S, and let $\boldsymbol{p}(q)$ denote the image of q on Σ under the normal mapping; then* (a) *ξ, η is $1-1$ in a neighborhood of q provided that the ω-component of the normal vector to Σ at $\boldsymbol{p}(q)$ does not vanish; and* (b)

$$\cos^2\vartheta/R^2 \leqq (d\xi/ds)^2 + (d\eta/ds)^2 \leqq 1/R^2 \sin^2\varphi, \tag{37}$$

where ds is the Euclidean element of length on S, ϑ is the angle between the tangent plane to Σ at $\boldsymbol{p}(q)$ and the σ, τ plane, and φ is the angle between the tangent plane to Σ at $\boldsymbol{p}(q)$ and the radial direction.

Proof of part (a). Let $\bar{V}$ be a coordinate patch of S containing q, and such that the plane of the local coordinates $\bar{x}, \bar{y}$ is parallel to the tangent plane to S at q. Let $\bar{U}$ be a coordinate patch of Σ, with local coordinates $\bar{u}, \bar{v}$, to which $\bar{V}$ corresponds. It follows that $\partial(\xi, \eta)/\partial(\bar{x}, \bar{y}) = \bar{F}^4[\partial(\sigma, \tau)/\partial(\bar{u}, \bar{v})] \neq 0$; since $\partial(\sigma, \tau)/\partial(\bar{u}, \bar{v})$ is the ω-component of $\boldsymbol{R}_{\bar{u}} \times \boldsymbol{R}_{\bar{v}}$. $\boldsymbol{R}$ again denotes the position vector of points on Σ.

Proof of part (b). If f, g are quadratic forms on Σ, it is clear from the definition that in corresponding coordinate patches $\bar{V}$, $\bar{U}$ of S and Σ, respectively, the ratio $\Phi f/\Phi g$ at a given point of $\bar{V}$ in the direction $d\bar{y}/d\bar{x} = m$ has the same value as the ratio f/g at the normal image point on $\bar{U}$ in the direction $d\bar{v}/d\bar{u} = -1/m$. The result now follows since $d\xi^2 + d\eta^2 = \Phi(d\sigma^2 + d\tau^2)$, $ds^2 = \Phi(\boldsymbol{R} \times d\boldsymbol{R})^2$, and clearly

$$\cos^2\vartheta/R^2 \leqq (d\sigma^2 + d\tau^2)/(R \times dR)^2 \leqq 1/R^2 \sin^2\varphi. \tag{38}$$

In a neighborhood of a point p of Σ at which the normal vector has a non-vanishing ω-component, the ω-coordinate on Σ may be taken to be a function of σ, τ. According to the preceding lemma, in a neighborhood of a point q, whose normal image on Σ is p, ξ, η may be taken as independent variables. Parts (a) and (b) of the following lemma are due to RADÒ.

Lemma 4. *In a neighborhood of a point q, where the ξ, η mapping is 1—1;* (a) $z_\xi = \sigma$, $z_\eta = \tau$; (b) *$z(\xi, \eta)$ is a solution of the Euler equation of the non-parametric variational problem $\delta \int \omega(\sigma, \tau)\, d\xi\, d\eta = 0$; and* (c) $K(q) = \frac{1}{R^4} \partial(\sigma, \tau)/\partial(\xi, \eta)$, *where $K(q)$ is the Gauss curvature of S at q.*

Proof of part (a). Observe that in any coordinate patch $\overline{V}$ of S, which corresponds to a coordinate patch $\overline{U}$ of Σ; $d\eta = 0$ in the direction $d\bar{y}/d\bar{x} = \sigma_{\bar{v}}/\sigma_{\bar{u}}$, and that $d\xi = 0$ in the direction $dy/dx = \tau_{\bar{v}}/\tau_{\bar{u}}$. The result now follows from Lemma 2.

Proof of part (b). To prove part (b), note that the Euler equation of the variational problem involved may be written in the form $(\omega_\sigma)_\xi + (\omega_\tau)_\eta = 0$. The result now follows from Lemma 2; since in any coordinate patch $\overline{V}$ of S, corresponding to a coordinate patch $\overline{U}$; of Σ,

$$\omega_\tau\, d\xi - \omega_\sigma\, d\eta = \overline{F}^2(\omega_{\bar{v}}\, d\bar{x} - \omega_{\bar{u}}\, d\bar{y}) = d\psi. \tag{39}$$

Proof of part (c). Let $\overline{V}$ be a coordinate patch of S containing q and corresponding to a coordinate patch $\overline{U}$ of Σ, then

$$K(q) = \left(\frac{1}{W}\right)^4 \frac{\partial(\bar{u}, \bar{v})}{\partial(\bar{x}, \bar{y})} = \left(\frac{1}{W}\right)^4 \left[\frac{\partial(\xi, \eta)}{\partial(\bar{x}, \bar{y})} \Big/ \frac{\partial(\sigma, \tau)}{\partial(\bar{u}, \bar{v})}\right] \frac{\partial(\sigma, \tau)}{\partial(\xi, \eta)}, \tag{40}$$

but

$$\left(\frac{1}{W}\right)^4 \left[\frac{\partial(\xi, \eta)}{\partial(\bar{x}, \bar{y})} \Big/ \frac{\partial(\sigma, \tau)}{\partial(\bar{u}, \bar{v})}\right] = \left(\frac{\overline{F}}{W}\right)^4 = \frac{1}{R^4}. \tag{41}$$

In the case of non-parametric variational problems of the type $\delta\int F(u, v)\, dx\, dx = 0$ the addition of a linear function of u, v to the integrand $F(u, v)$ leaves the Euler equation and hence also the class of solutions unchanged. An analogous situation exists in the case of the parametric variational problems under consideration. Since the integrand $\mathcal{J}(\boldsymbol{p})$ of the variational problem (1) is completely defined by its values on the unit vectors, $\mathcal{J}(\boldsymbol{p})$ is uniquely determined by the surface Σ. We now show that Σ is not uniquely associated with the class of solution surfaces but may be chosen according to the following lemma.

Lemma 5. *Let P be a projective transformation of the space Y which maps each line through the origin onto itself, preserving orientation. Assume also that P maps every point of Σ to a finite point. Let Σ^* denote the images of Σ under the transformation P; i.e., $\Sigma^* = P\Sigma$. The conclusion is that the class of solutions is unchanged if Σ is replaced with Σ^*.* *

Proof. Let σ, τ, ω be an arbitrary coordinate system for Y, and let U, U^* be the coordinate patches of Σ, Σ^* respectively for which $\omega < 0$. The hypotheses on P imply that $PU = U^*$ and that points of U, U^* which correspond under P

* It can be readily verified that replacing Σ by Σ^* has the effect of adding linear functions to the integrands of the local non-parametric variational problems.

have the same local coordinates u, v. Let f, f^* denote the second fundamental forms of Σ, Σ^* respectively; then in U according to Lemma 1, $f = \chi(u, v) f^*$. Let S be a solution surface of the problem associated with Σ; then by Theorem 1, $f/\Phi f = -R^4 K$. Let V be a coordinate patch of S corresponding to U, and hence also to U^*, and let F^*, R^* be the quantities associated with Σ^* which are analogous to F, R (*i.e.*, $F^*(u, v)$ is the negative reciprocal of the ω-coordinate of the point of U^* whose local coordinates are u, v); then since $F/F^* = R^*/R$,

$$f^*/\Phi f^* = (F^4/F^{*4})\,(f/\Phi f) = -R^{*4} K. \tag{42}$$

The hypotheses of Lemma 5 can be weakened in the following sense. Let $\mathscr{N}$ be a subset of Σ, and let P be a projective transformation which satisfies the hypotheses of Lemma 5, except that instead of requiring that P map all points of Σ to finite points, we require only that P maps all points of $\Sigma - \mathscr{N}$ to finite points. The class of solution surfaces whose images under the normal mapping omits the set $\mathscr{N}$ of Σ is then unchanged if Σ is replaced with $\Sigma^* = P\Sigma$.

VI. In Sections II and IV we have assumed that $\mathscr{J}(\boldsymbol{p})$ is positive homogeneous and positive definite. We now assume in addition that $\mathscr{J}(\boldsymbol{p})$ is regular; *i.e.*, Σ is a closed surface of positive curvature.

Lemma 6. *Let x, y, z be a fixed coordinate system for X, and let $\mathscr{S}$ denote the class of solution surfaces of the variational problem associated with Σ whose normal directions omit the positive z direction. Assume that Σ is a surface of positive curvature and hence is convex. Let σ, τ, ω be a coordinate system for Y, with axes parallel respectively to the x, y, z axes. Let p_0 be the point of intersection of Σ with the positive ω-axis. There then exists a projective transformation P with the following properties:* (a) *P maps p_0 to a point at infinity, P maps all points of $\Sigma - p_0$ to finite points, and otherwise P satisfies the hypotheses of Lemma 5;* (b) *the class $\mathscr{S}$ is unchanged if Σ is replaced with $\Sigma^* = P\Sigma$;* (c) *the surface $\Sigma^* = P\Sigma$ is representable in the form $\omega = G(\sigma, \tau)$ where G is defined for all σ, τ; and* (d) *the Euler equation of the variational problem $\delta \int G(\sigma, \tau)\, d\xi\, d\eta = 0$ is uniformly elliptic in the entire σ, τ plane.*

Corollary. *Let S be a simply connected solution surface whose normal directions at each point make an angle $\geq \beta$ with the positive z direction. Let $d\sigma$, $d\tau$ be the differentials of the coordinate functions on Σ^*. The functions ξ, η arising by integration from the exact differentials $d\xi = \Phi(d\tau)$, $d\eta = \Phi(-d\sigma)$ then define a mapping of S into the plane which is,* (a) *locally 1—1 everywhere on S; and* (b) *has the property that $(d\xi/ds)^2 + (d\eta/ds)^2 \geq c^2$ at every point of S, where ds is the Euclidean element of length on S, and c is a constant depending only on β and Σ.*

Proof of the corollary. The normal image of S lies in a compact subset of Σ^*. Σ^* however is a surface of positive curvature of the form $\omega = G(\sigma, \tau)$, where G is defined for all values of σ, τ. The inclination of the tangent plane to Σ^* is therefore bounded on compact subsets. The corollary is therefore an immediate consequence of Lemma 5, and parts (a) and (b) of Lemma 3.

Proof of Lemma 6. Let T denote the tangent plane to Σ at p_0, and let H denote the plane through the origin parallel to T. Let $\bar\sigma, \bar\tau, \bar\omega$ be a coordinate system for Y such that the $\bar\sigma, \bar\tau$ plane coincides with H, and T intersects the

positive $\overline{\omega}$-axis. We now define the projective transformation P in terms of the $\overline{\sigma}, \overline{\tau}, \overline{\omega}$ coordinates as follows:

$$\overline{\sigma}^* = \frac{\overline{\sigma}}{1-\overline{\omega}/h}, \quad \overline{\tau}^* = \frac{\overline{\tau}}{1-\overline{\omega}/h}, \quad \overline{\omega}^* = \frac{\overline{\omega}}{1-\overline{\omega}/h}, \tag{43}$$

where h is the perpendicular distance from the origin to the plane T. From (43) it follows immediately that P maps each line through the origin onto itself preserving orientation; that P maps p_0 to infinity, and all points of $\Sigma - p_0$ to finite points; and that P maps the plane T to infinity and leaves the plane H pointwise fixed. The equations (43) also imply that the image under P of a line L intersecting T in the point Q' and H in the point Q'', is the line L' through Q'' and parallel to the line through the origin and Q'. The surface $\Sigma^* = P\Sigma$ and also the function $G(\sigma, \tau)$ can now be constructed in the fashion of projective geometry (see Fig. 1). Let Q be an arbitrary point in the σ, τ plane, and let L_1 be the line through Q, parallel to the ω-axis. Let Q_1 be the point where L_1 intersects the plane H, and let L_2 denote the line through Q_1 and p_0.

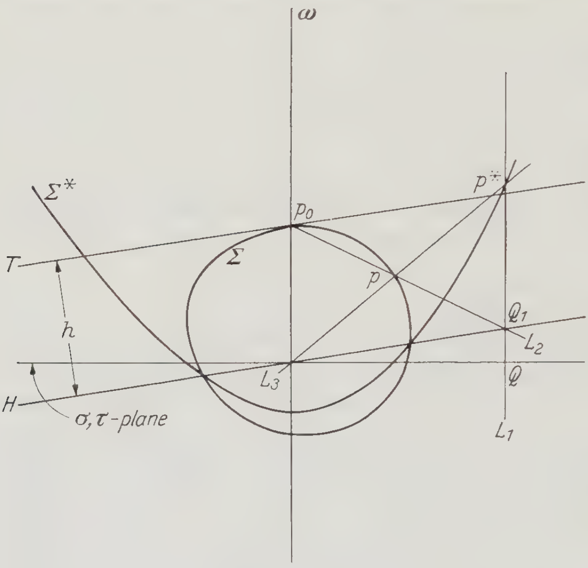

Fig. 1. Construction of Σ^*

According to the above remarks, then, the line L_2 is mapped onto the line L_1 under the transformation P. Let p be the point where the segment $\overline{p_0 Q_1}$ intersects Σ, and let L_3 be the line through the origin and p. Since L_2 is mapped onto L_1 by P, and L_3 is mapped onto itself, it follows that the image of p under P is the point p^* in which L_3 intersects L_1. If σ, τ are the coordinates of Q, then $G(\sigma, \tau)$ is the height of p^* above the σ, τ plane. This proves parts (a) and (c) of Lemma 6, and part (b) follows directly from Lemma 5.

Proof of part (d). We first observe that the ellipticity of the Euler equation of the variational problem $\delta \int G(\sigma, \tau)\, d\xi\, d\eta = 0$ is equivalent to the condition that Σ^* is a surface of positive curvature and hence follows from Lemma 1. The condition of uniform ellipticity is equivalent to the condition that the second fundamental form of Σ^*, considered as a Riemannian metric over the σ, τ plane, is quasi-conformally related to the metric $d\sigma^2 + d\tau^2$. According to equation (28) this will be the case if and only if there is a constant k such that

$$\frac{G_{\sigma\sigma} + G_{\tau\tau}}{(G_{\sigma\sigma} G_{\tau\tau} - G_{\sigma\tau}^2)^{\frac{1}{2}}} < k < \infty. \tag{44}$$

We note next that if $\boldsymbol{R}$ denotes the position vector of a surface and if the surface has the property that the angle between the outer normal vector and $\boldsymbol{R}$ is acute at each point, then $(\boldsymbol{R}\times d\boldsymbol{R})^2$ is a positive definite quadratic form and may be taken as a Riemannian metric on the surface. Furthermore, if $\boldsymbol{R}^*$ is the position vector of another such surface, then the mapping of one surface on the other by projection through the origin is conformal in the metrics $(\boldsymbol{R}\times d\boldsymbol{R})^2$, $(\boldsymbol{R}^*\times d\boldsymbol{R}^*)^2$. This follows since such a mapping has the form $\boldsymbol{R}^*=\lambda\boldsymbol{R}$ where λ is a scalar function, and hence $\boldsymbol{R}^*\times d\boldsymbol{R}^*=\lambda\boldsymbol{R}\times\big(\lambda\, d\boldsymbol{R}+(d\lambda)\,\boldsymbol{R}\big)=\lambda^2(\boldsymbol{R}\times d\boldsymbol{R})$. Now let $\boldsymbol{R}$, $\boldsymbol{R}^*$ denote the position vectors of Σ, Σ^* respectively. We shall show that the metric $(\boldsymbol{R}^*\times d\boldsymbol{R}^*)^2$ is quasi-conformally related to the metric $d\sigma^2+d\tau^2$. Part (d) of Lemma 6 then follows from the transitivity of the relation of quasi-conformality. To see this, observe that by the above remarks, the transformation P maps Σ^* conformally onto $\Sigma-p_0$ with respect to the metrics $(\boldsymbol{R}^*\times d\boldsymbol{R}^*)^2$, $(\boldsymbol{R}\times d\boldsymbol{R})^2$. The metric $(\boldsymbol{R}\times d\boldsymbol{R})^2$ on Σ is necessarily quasi-conformally related to the second fundamental form of Σ considered as a metric, by the compactness of Σ. On the other hand, Lemma 1 implies that the transformation P is conformal with respect to the second fundamental forms considered as metrics on Σ, Σ^*. It remains to show that the metric $(\boldsymbol{R}^*\times d\boldsymbol{R}^*)^2$ is quasi-conformally related to the metric $d\sigma^2+d\tau^2$. $\boldsymbol{R}^*=\big(\sigma,\tau,G(\sigma,\tau)\big)$, and it now follows from a calculation using equation (28) that what must be shown is that

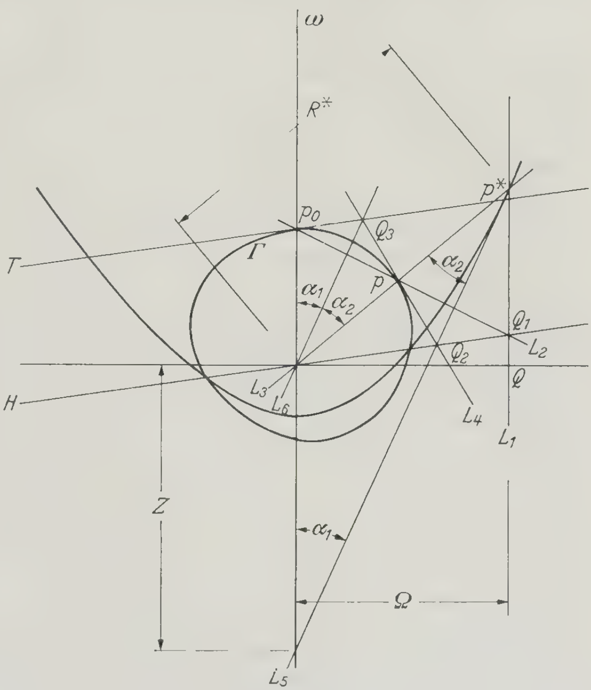

Fig. 2

$$(45)\qquad \frac{R^{*2}+(G-\sigma G_\sigma-\tau G_\tau)^2+(\sigma G_\tau-\tau G_\sigma)^2}{R^*\,|G-\sigma G_\sigma-\tau G_\tau|}<k<\infty,$$

where $R^*=|\boldsymbol{R}^*|$. Let $\Omega=(\sigma^2+\tau^2)^{\frac{1}{2}}$, and put $Z=|G-\sigma G_\sigma-\tau G_\tau|=|G-\Omega G_\Omega|$. Let L_1, L_2, L_3 be defined as in Fig. 1, and let ψ denote the plane containing the lines L_1, L_2, L_3. Let L_4 be the tangent line to Σ at p, in the plane ψ (see Fig. 2). Let Q_2, Q_3 be the points where L_4 intersects the plane H and T respectively. Let L_5 denote the line through Q_2 and p^*; then under the transformation P, L_4 is mapped onto L_5. L_5 is therefore the tangent line to Σ^* at p^* in the plane ψ. Since ψ contains the ω-axis, the slope of L_5 is the quantitiy G_Ω. It follows that the quantitiy $Z=|G-\Omega G_\Omega|$ is equal to the distance from the origin to the

point where L_5 intersects the negative ω-axis. Let α_1 be the angle between L_3 and the ω-axis, and let α_2 be the angle between L_3 and L_5. Application of the law of sines shows that $Z/R^* = \sin\alpha_2/\sin\alpha_1$. The line L_5 is parallel to the line L_6 through the origin and Q_3; therefore α_1 is equal to the angle between L_3 and L_6. Let Γ be the curve of intersection of Σ and the plane ψ. It may be shown by elementary means that the limit as p approaches p_0 along Γ of $\sin\alpha_1/\sin\alpha_2$ is equal to one less than the order of contact of Γ with its tangent line at p_0. Since the curvature of Γ does not vanish, the order of contact is 2, hence the limit in question is 1. From the smoothness properties of Σ it may now be inferred that as a function of a point p on Σ, the quantity Z/R^* is continuous and positive. Thus there exist positive constants k_1, k_2 such that $k_1 < Z/R^* < k_2$. The left-hand side of (45) may be written in the form $R^*/Z + Z/R^* + (\sigma G_\tau - \tau G_\sigma)^2/(R^* Z)$, and the boundedness of the first two terms has now been established. It is sufficient to show that the last term is bounded in the exterior of a compact region of the σ, τ plane. Let Γ_h denote the level curve of Σ^* at height h. Since Σ^* contains the origin, for all $h \geqq 0$, Γ_h is a closed convex curve enclosing the ω-axis. The curve Γ_h is in fact the projection through the origin of a certain plane section of Σ^* onto the horizontal plane at height h. It may be verified easily that the level curve at infinite height is the projection through the origin of the Dupin indicatrix of Σ at p_0 onto the horizontal plane at infinity and hence is an ellipse of finite eccentricity with center on the ω-axis. It now follows that the angle δ between ∇G and the radial direction in the σ, τ plane is bounded from $\frac{1}{2}\pi$ in the exterior of Γ_0; *i.e.*, there is an angle δ_0 such that $0 \leqq \delta \leqq \delta_0 < \frac{1}{2}\pi$ in the exterior of Γ_0. In the exterior of Γ_0, therefore, $|\sigma G_\tau - \tau G_\sigma| = \Omega |\nabla G| \sin\delta = \Omega G_\Omega \tan\delta < \Omega G_\Omega \tan\delta_0$, and hence

$$\frac{(\sigma G_\tau - \tau G_\sigma)^2}{R^* Z} < \frac{(\Omega G_\Omega \tan\delta_0)^2}{R^* Z} \leqq \frac{(R^* + Z)^2 \tan^2\delta_0}{R^* Z}. \tag{46}$$

Lemma 7. *Let S be a simply connected surface of class C^2. Let q be a point of S, and let d be the distance along S from q to the boundary of S. Let ξ, η be differentiable functions on S. Assume that* (i) *the mapping defined by the functions ξ, η of S into the plane is locally $1-1$ at all points of S and that* (ii) $(d\xi/ds)^2 + (d\eta/ds)^2 \geqq c^2$ *where ds is the Euclidean element of length on S and c is a positive constant. The conclusion is that the functions ξ, η map a subregion of the geodesic circle of radius d about q on S homeomorphically onto the circle $(\xi - \xi(q))^2 + (\eta - \eta(q))^2 < (cd)^2$. If S is complete, it follows that the functions ξ, η map S homeomorphically onto the entire ξ, η plane.*

Proof. Let r be the radius of the largest circle in the ξ, η plane about the point $(\xi(q), \eta(q))$ in which the inverse mapping is $1-1$. Suppose that $r < cd$. Let l denote the pre-image of any given radius of the circle $(\xi - \xi(q))^2 + (\eta - \eta(q))^2 < r^2$. The length of l is then necessarily $< d$ since

$$r = \int_l \left((d\xi/ds)^2 + (d\eta/ds)^2\right)^{\frac{1}{2}} ds \geqq c \cdot \text{length of } l. \tag{47}$$

Since the mapping is locally $1-1$, the inverse mapping is therefore $1-1$ in a neighborhood of every boundary point of the circle $(\xi - \xi(q))^2 + (\eta - \eta(q))^2 < r^2$ and hence is $1-1$ in some circle of radius $> r$. A different proof, which is adaptable to the present case, is given in [4] for the case in which S is a plane region.

Let $ds^2 = a\,d\sigma^2 + 2b\,d\sigma\,d\tau + c\,d\tau^2$ be a Riemannian metric defined over a region D of the σ, τ plane. Let $\varrho = \sigma + i\tau$, $\chi(\varrho) = p(\sigma, \tau) + iq(\sigma, \tau)$; then the mapping $\chi(\varrho)$ is a conformal mapping of the Riemannian manifold defined over D by the metric ds^2 provided p, q is a solution of the Beltrami system with coefficients a, b, c. This Beltrami system is given in complex form by

$$\chi_{\bar{\varrho}} = \mu(\varrho)\,\chi_{\varrho}, \tag{48}$$

where $\mu(\varrho) = (a - c + 2ib)/\left(a + c + 2(ac - b^2)^{\frac{1}{2}}\right)$. The Beltrami equation (48) is uniformly elliptic if there is a constant k such that $(a+c)/(ac-b^2)^{\frac{1}{2}} < k < \infty$ or equivalently if there is a constant μ_0 such that $0 < |\mu| \leqq \mu_0 < 1$.

Lemma 8. *Let $\chi(\varrho)$ be a solution of the Beltrami system* (48) *in the disc $|\varrho| < 1$. Assume that $|\chi| < M$, $|\mu| \leqq \mu_0 < 1$ and that $\mu(\varrho)$ satisfies the uniform Hölder condition $|\mu(\varrho_1) - \mu(\varrho_2| \leqq H|\varrho_1 - \varrho_2|^{\alpha}$. Let ϱ be an interior point of the disc $|\varrho| < 1$. Let d_{ϱ} be the distance from ϱ to the boundary. Then there is a constant $k(\alpha, H, \mu_0)$, depending only on the quantities indicated, such that $d_{\varrho}|\chi_{\varrho}(\varrho)| < k(\alpha, H, \mu_0)M$.* This lemma is a special case of a theorem of Douglis & Nirenberg [*13*] on Schauder estimates for elliptic systems. A proof for Beltrami systems is given in [*4*].

At a point where the Jacobian does not vanish, a differentiable mapping carries infinitesimal circles into infinitesimal ellipses. If δ is the ratio of the minor axis to the major axis at a given point, then the eccentricity E of the mapping at the point is defined to be $E = \frac{1}{2}(\delta + 1/\delta)$. A mapping which is continuously differentiable in a domain D is defined to be quasi-conformal in D if its eccentricity is uniformly bounded there. For a mapping $u(x,y)$, $v(x,y)$, this definition is equivalent to the condition that the Jacobian $\partial(u,v)/\partial(x,y)$ vanishes only at isolated points, and that the metric $du^2 + dv^2$ is quasi-conformally related to the metric $dx^2 + dy^2$. It can be shown that the eccentricity of a mapping given by a solution of the Beltrami system $\chi_{\varrho} = \mu(\varrho)\chi_{\varrho}$ is given by $E = \left(1 + |\mu|^2\right)/\left(1 - |\mu|^2\right)$. Thus such a mapping is quasi-conformal if the Beltrami system is uniformly elliptic. If $\varphi(x,y)$ is a solution of an elliptic equation $a\varphi_{xx} + 2b\varphi_{xy} + c\varphi_{yy} = 0$, $ac - b^2 > 0$, it can be shown that the eccentricity E of the mapping $u(x,y)$, $v(x,y)$, where $u = \varphi_x$, $v = \varphi_y$, satisfies an inequality $E \leqq \frac{1}{2}\left((a+c)^2/(ac-b^2) - 1\right)$. Such a mapping is therefore quasi-conformal if the equation satisfied by $\varphi(x,y)$ is uniformly elliptic.

Lemma 9. *Let $w = u + iv$ be continuously differentiable with eccentricity $\leqq E_0$ in a domain A of; the z-plane. Assume that $|w| \leqq 1$ then in any compact subregion B of A, $w(z)$ satisfies a uniform Hölder condition*

$$|w(z_1) - w(z_2)| \leqq H\left|\frac{z_1 - z_2}{d}\right|^{\delta}, \tag{49}$$

where $\delta = E_0 - (E_0^2 - 1)^{\frac{1}{2}}$, H is an absolute constant, and d is the distance from B to the boundary of A.

Proofs of the Hölder continuity of quasi-conformal mappings have been given by many authors. For further discussion, and a proof of Lemma 9 with $H = \pi e$, see [*14*].

The following lemma is due to Warschawski [*15*].

Lemma 10. *Let R be a region in the z-plane bounded by a continuously differentiable closed Jordan curve C. Assume that as a function of arc length on C, the*

angle ϑ, between the tangent line to C and the x-direction, is Hölder continuous with constant k and exponent ν. Let D be the diameter of C. Let z_0 be an interior point of R, and let ϱ be the radius of the largest circle about z_0 lying entirely in R. Let $d = GLB(r/\sigma)$, where r is the distance between any two points of C, and σ is the length of the shorter arc of C connecting them. Let $w = f(z)$ be the Riemann mapping function which maps R conformally and homeomorphically onto the disc $|w| < 1$, with $f(z_0) = 0$. The conclusion is that there are constants μ_1, μ_2 such that $0 < \mu_1 \leqq |f'(z)| \leqq \mu_2$, for all z in R, where μ_1, μ_2 depend only on k, ν, D, ϱ, d.

We now establish the inequalities mentioned in the Introduction.

Theorem 3. *Let S be a simply connected solution surface of the variational problem $\delta \int \mathscr{J}(\mathbf{p})\, ds\, dt = 0$. Assume that $\mathscr{J}(\mathbf{p}) \in C^{2+\alpha}$ for all $\mathbf{p} \neq 0$, and that $\mathscr{J}(\mathbf{p})$ is positive definite, positive homogeneous of degree 1, and regular. Assume that the normal vector at each point of S makes an angle $\geqq \beta$ with the positive z-axis. Let q be a point of S, and d be the distance along S from q to the boundary. Let α be the angle between the normal at q and the positive z-direction. Then*

$$|K(q) \leqq C(\beta) \frac{(\alpha-\beta)^2}{d^2}, \tag{50}$$

where $K(q)$ is the Gaussian curvature of S at q, and $C(\beta)$ is a constant which depends only on β and the integrand $\mathscr{J}(\mathbf{p})$. If S is a surface of the form $z = \varphi(x, y)$, then

$$|K(x_0, y_0)| \leqq \frac{C_1}{w_0^2 r^2}, \tag{51}$$

and

$$r_0^2 + 2 s_0^2 + t_0^2 \leqq \frac{C_2 W_0^4}{r^2}, \tag{52}$$

where $K(x_0, y_0)$ is the Gaussian curvature of S at x_0, y_0, r is the radius of the largest circle about x_0, y_0 in which $\varphi(x, y)$ is defined, $W_0 = \left(1 + \varphi_x^2(x_0, y_0) + \varphi_y^2(x_0, y_0)\right)^{\frac{1}{2}}$, and r_0, s_0, t_0 are the derivatives $\varphi_{xx}(x_0, y_0)$, $\varphi_{xy}(x_0, y_0)$, $\varphi_{yy}(x_0, y_0)$ respectively.

Corollary. *Under the same hypotheses, if S is complete, then S is a plane.*

Proof. Let Σ^* be the surface constructed according to Lemma 6. The right circular cone having vertex at the origin and generators making an angle β with the positive ω-axis intersects Σ^* in a closed curve $\Gamma^*_{(\beta)}$ which bounds a compact region $D^*(\beta)$ on Σ^*. By assumption, the image of S on Σ^* under the normal mapping lies in $D^*(\beta)$. Let $D(\beta)$ be the projection of $D^*(\beta)$ onto the σ, τ plane. Our smoothness assumptions on Σ imply that the function $G(\sigma, \tau)$ is of class $C^{2+\alpha}$, uniformly so on compact subsets. Thus it follows that the boundary $\Gamma(\beta)$ of $D(\beta)$ is a closed convex curve of class $C^{2+\alpha}$. Let ξ, η be the functions on S arising from the differentials of the coordinate functions on Σ^* by means of the Φ-mapping and Lemmas 2, 5 and 6. It may be assumed that $\xi(q) = \eta(q) = 0$. By Lemmas 6 and 7, the functions ξ, η map a subregion of the geodesic circle on S of radius d about q homeomorphically onto the circle $\xi^2 + \eta^2 < c^2 d^2$, where c is the constant of the corollary to Lemma 6. According to Lemmas 4 and 6, in $\xi^2 + \eta^2 < c^2 d^2$, $z_\xi = \sigma$, $z_\eta = \tau$, and $z(\xi, \eta)$ is a solution of the Euler equation of the non-parametric variational problem $\delta \int G(\sigma, \tau)\, d\xi\, d\eta = 0$. Thus $z(\xi, \eta)$ satisfies the equation

$$G_{\sigma\sigma} z_{\xi\xi} + 2 G_{\sigma\tau} z_{\xi\eta} + G_{\tau\tau} z_{\eta\eta} = 0. \tag{53}$$

By part (d) of Lemma 6, this equation is uniformly elliptic, and according to the remarks made above, the values assumed by the derivatives $z_\xi = \sigma$, $z_\eta = \tau$ lie in the region $D(\beta)$.

The corollary follows, since if S is a complete surface, by Lemma 7 the ξ, η mapping is a $1-1$ mapping of S onto the plane. $\varrho(\xi) = \sigma(\xi, \eta) - i\tau(\xi, \eta)$ is then bounded and quasi-formal in the entire plane. Since Liouville's theorem is valid for quasi-conformal mappings, it follows that $\varrho(\zeta)$ is a constant. The normal image of S on Σ^* is therefore a single point, hence S is a plane.

By part (c) of Lemma 4 and Lemma 6, the Gaussian curvature $K(q)$ of S at q is given by

$$K(q) = (1/R^{*4}) \left.\frac{\partial(\sigma, \tau)}{\partial(\xi, \eta)}\right|_{\xi=\eta=0}. \tag{54}$$

Let $H(G)$ denote the Riemannian manifold defined over the σ, τ plane by the metric $ds_1^2 = G_{\sigma\sigma} d\sigma^2 + 2G_{\sigma\tau} d\sigma\, d\tau + G_{\tau\tau} d\tau^2$. Let $K\big(G, z(\xi, \eta)\big)$ denote the Riemannian manifold defined over the disc $\xi^2 + \eta^2 < c^2 d^2$ by the metric $ds_2^2 = G_{\tau\tau} d\xi^2 - 2G_{\sigma\tau} d\xi\, d\eta + G_{\sigma\sigma} d\eta^2$. A calculation based on equation (53) shows that

$$G_{\sigma\sigma} d\sigma^2 + 2G_{\sigma\tau} d\sigma\, d\tau + G_{\tau\tau} d\tau^2 = (z_{\xi\eta}^2 - z_{\xi\xi} z_{\eta\eta})(G_{\tau\tau} d\xi^2 - 2G_{\sigma\tau} d\xi\, d\eta + G_{\sigma\sigma} d\eta^2). \tag{55}$$

It is well known that $z_{\eta\xi}^2 - z_{\xi\xi} z_{\eta\eta}$ vanishes only at isolated points if $z(\xi, \eta)$ is a solution of equation (53); hence the function $\varrho(\zeta) = \sigma(\xi, \eta) - i\tau(\xi, \eta)$ maps $K\big(G, z(\xi, \eta)\big)$ conformally into $H(G)$, where we have put $\zeta = \xi + i\eta$.

In the minimal surface case, Σ is the unit sphere, and Σ^* is the paraboloid of revolution given by $\omega = \frac{1}{2}(\sigma^2 + \tau^2 - 1)$. Equation (53) is then the Laplace equation, and the projective mapping P of Σ into Σ^* followed by the parallel projection of Σ^* onto the plane is just the stereographic projection of Σ onto the σ, τ plane. The region $D(\beta)$ is then the circle of radius $\cot \frac{1}{2}\beta$ about the origin, and $\varrho(\zeta)$ is a complex analytic function with values in $D(\beta)$. $\partial(\sigma, \tau)/\partial(\xi, \eta)$ can then be estimated by means of Schwarz's lemma, and the estimate of the curvature follows from equation (54). This procedure closely parallels OSSERMAN's proof for minimal surfaces.

In the general case, in order to estimate $\partial(\sigma, \tau)/\partial(\xi, \eta)$, we introduce uniformizing variables $\chi(\varrho)$, $\gamma(\zeta)$ for the manifolds $H(G)$, $K\big(G, z(\xi, \eta)\big)$ respectively. The coefficients of the metric ds_1 are Hölder-continuous; hence by the general uniformization theorem there is a conformal homeomorphism $t(\varrho)$, with Hölder-continuous first derivatives, non-vanishing Jacobian, and such that $t(0) = 0$, which maps $H(G)$ onto either the disc $|t| < 1$, or the entire t-plane. It must be the latter, however, since the uniform ellipticity of equation (53) implies that the metric ds_1 is quasi-conformally related to the metric $d\sigma^2 + d\tau^2$, and the relation of quasi-conformality preserves conformal type. Under the mapping $t(\varrho)$, the region $D(\beta)$ is mapped onto a region $D'(\beta)$ in the t-plane. The smoothness properties of the curve $\Gamma(\beta)$ and the mapping $t(\varrho)$ imply that the boundary $\Gamma'(\beta)$ of $D'(\beta)$ is a closed curve of class $C^{1+\alpha}$. Let $\chi(t)$ be the Riemann mapping function which maps $D'(\beta)$ conformally onto the unit circle $|\chi| < 1$, with $\chi(0) = 0$. According to Lemma 10, then, there exist two positive constants μ_1, μ_2 which depend only on the Hölder continuity and the geometry of the boundary curve $\Gamma'(\beta)$,

and hence only on the angle β and $\mathcal{J}(p)$, such that for all $t\in D'(\beta)$,

$$\mu_1 \leqq |(d\chi/dt)| \leqq \mu_2. \tag{56}$$

$\chi(\varrho)$ is a conformal mapping of $H(G)$; hence $\chi(\varrho)$ satisfies the Beltrami system $\chi_{\bar\varrho} = \mu(\varrho)\chi_\varrho$, where

$$\mu(\varrho) = \frac{(G_{\sigma\sigma} - G_{\tau\tau} + 2iG_{\sigma\tau})}{(G_{\sigma\sigma}+G_{\tau\tau}+2(G_{\sigma\sigma}G_{\tau\tau}-G_{\sigma\tau}^2)^{\frac{1}{2}})}. \tag{57}$$

Since equation (53) is uniformly elliptic, $\chi(\varrho)$ is quasi-conformal with eccentricity $\leqq E_0 = \operatorname{Sup}\big((G_{\sigma\sigma}+G_{\tau\tau})/(2(G_{\sigma\sigma}G_{\tau\tau}-G_{\sigma\tau}^2)^{\frac{1}{2}})\big)$, and $|\mu(\varrho)| \leqq \mu_0 = (E_0-1)^{\frac{1}{2}}/(E_0+1)^{\frac{1}{2}} < 1$. Since $\partial(t,\bar t)/\partial(\varrho,\bar\varrho)$ does not vanish in the entire σ, τ plane, it is bounded above and below by positive constants on the compact set $D(\beta)$.

Since

$$\frac{\partial(\chi,\bar\chi)}{\partial(\varrho,\bar\varrho)} = |\chi_\varrho|^2\left(1-|\mu(\varrho)|^2\right) = \frac{\partial(\chi,\bar\chi)}{\partial(t,\bar t)}\,\frac{\partial(t,\bar t)}{\partial(\varrho,\bar\varrho)}, \tag{58}$$

$$\left|\frac{d\chi}{ds}\right|^2 \leqq 2\left(|\chi_\varrho|^2+|\chi_{\bar\varrho}|^2\right) = 2|\chi_\varrho|^2\left(1+|\mu(\varrho)|^2\right), \tag{59}$$

where ds is the Euclidean element of length in the σ, τ plane, it follows that there are positive constants m_1, m_2, n_1, n_2 such that at all points of $D(\beta)$,

$$m_1 \leqq |\chi_\varrho| \leqq m_2, \qquad n_1 \leqq \left|\frac{d\chi}{ds}\right| \leqq n_2. \tag{60}$$

A conformal mapping $\gamma(\zeta)$ of $K(G, z(\xi,\eta))$ must satisfy the Beltrami system $\gamma_{\bar\zeta} = -\mu(\varrho(\zeta))\gamma_\zeta$. As previously noted, the second derivatives of $G(\sigma,\tau)$, and hence the Beltrami coefficient $\mu(\varrho)$ are in class C^α, and uniformly so on compact sets. Thus let H_1 be the Hölder constant of $\mu(\varrho)$ on $D(\beta)$. By the remarks preceding Lemma 9, $\varrho(\zeta)$ is quasi-conformal with eccentricity $\leqq E_1$, where

$$E_1 = \operatorname{Sup}\left(\frac{(G_{\sigma\sigma}+G_{\tau\tau})^2}{2(G_{\sigma\sigma}G_{\tau\tau}-G_{\sigma\tau}^2)} - 1\right) = 2E_0^2 - 1. \tag{61}$$

Let $M = \max\limits_{(\sigma,\tau)\in D(\beta)} (\sigma^2+\tau^2)^{\frac{1}{2}}$; then $|\varrho(\zeta)| \leqq M$. The Hölder continuity of the mapping $\varrho(\zeta)$ can then be estimated in the disc $|\zeta| \leqq \frac{1}{2}cd$ by means of Lemma 9. Combining the estimate thus obtained with the Hölder continuity of $\mu(\varrho)$, we see that in the disc $|\zeta| < \frac{1}{2}cd$, $\mu(\varrho(\zeta))$ satisfies a Hölder condition

$$|\mu(\varrho(\zeta_1)) - \mu(\varrho(\zeta_2))| \leqq H_1(HM)^\alpha \left|\frac{\zeta_1-\zeta_2}{\frac{1}{2}cd}\right|^{\alpha\delta_1}, \tag{62}$$

where $\delta_1 = E_1 - (E_1^2-1)^{\frac{1}{2}}$, and H is the absolute constant of Lemma 9. By the general uniformization theorem, there is a conformal homeomorphism $\gamma(\zeta)$ of the submanifold of $K(G, z(\xi,\eta))$ defined by $|\zeta| < \frac{1}{2}cd$, onto the disc $|\gamma| < \frac{1}{2}cd$ such that $\gamma(0) = 0$. We now show that

$$|\gamma_\zeta(0)| \leqq K_1, \tag{63}$$

where K_1 is a constant depending only on the angle β and $\mathcal{J}(\boldsymbol{p})$. To see this, put $\gamma^* = (2\gamma)/(cd)$, $\zeta^* = (2\zeta)/(cd)$, and $\mu^*(\zeta^*) = -\mu\left(\varrho\left(\frac{1}{2}cd\,\zeta^*\right)\right)$. In $|\zeta^*| < 1$, then,

$|\gamma^*| < 1$, $\mu^*(\zeta^*)$ is uniformly Hölder continuous with constant $(HM)^\alpha H_1$ and exponent $\alpha\delta_1$, and $\gamma^*(\zeta^*)$ satisfies the Beltrami system $\gamma_{\bar{\zeta}^*} = \mu^*(\zeta^*)\gamma_{\zeta^*}$. Since $\gamma_\zeta(\zeta) = \gamma^*_{\zeta^*}(\zeta^*)$, the result now follows from Lemma 8.

It follows from the definitions of $\chi(\varrho)$ and $\gamma(\zeta)$ that χ is a complex analytic function of γ in the disc $|\gamma| < \frac{1}{2}cd$. $\chi_\gamma(\gamma)$ can therefore be estimated at the origin by applying Schwarz's lemma to the function $(\chi - \chi_0)/(1 - \chi\bar{\chi}_0)$, where $\varrho_0 = \sigma(0,0) - i\tau(0,0)$, and $\chi_0 = \chi(\varrho_0)$. The result is

$$|\chi_\gamma(0)| \leqq \frac{1 - |\chi_0|^2}{\frac{1}{2}cd}. \tag{64}$$

Now

$$\frac{\partial(\sigma,\tau)}{\partial(\xi,\eta)} = \frac{\partial(\varrho,\bar{\varrho})}{\partial(\chi,\bar{\chi})}\,\frac{\partial(\chi,\bar{\chi})}{\partial(\gamma,\bar{\gamma})}\,\frac{\partial(\gamma,\bar{\gamma})}{\partial(\zeta,\bar{\zeta})}. \tag{65}$$

Since $\partial(\gamma,\bar{\gamma})/\partial(\zeta,\bar{\zeta}) = |\gamma_\zeta|^2(1 - |\mu(\varrho(\zeta))|^2)$, it follows from (58), (60), (63) and (64) that

$$\left|\frac{\partial(\sigma,\tau)}{\partial(\xi,\eta)}\right|_{\xi=\eta=0} = \frac{|\chi_\gamma(0)|^2\,|\gamma_\zeta(0)|^2}{|\chi_\varrho(\varrho_0)|^2} \leqq \left(\frac{2K_1}{m_1 c}\right)^2 \frac{(1 - |\chi_0|^2)^2}{d^2}. \tag{66}$$

Now, let ψ_1 denote the plane through the ω-axis and the point $(\sigma(0,0), \tau(0,0))$. Let L denote that segment of the line in which ψ_1 intersects the σ, τ plane which lies between the point $(\sigma(0,0), \tau(0,0))$ and the boundary of $D(\beta)$. Let b_0 denote the length of L. Under the mapping $\chi(\varrho)$, the segment L is mapped onto an arc connecting the point χ_0 with the boundary of the disc $|\chi| < 1$. Thus, letting s denote arc length in the σ, τ plane, and using (60), we see that

$$1 - |\chi_0| \leqq \int\limits_L \left|\frac{d\chi}{ds}\right| ds \leqq n_2 b_0. \tag{67}$$

Let L^* denote the arc of the curve in which ψ_1 intersects Σ^* which lies over L. Let φ be the angle between the line from the origin to a point of L^* and the positive ω-axis. It follows by calculation that $|ds/d\varphi| = R^*(R^*/(\Omega G_\Omega - G))$. In the proof of part (d) of Lemma 6, it was established that $k_1 \leqq (\Omega G_\Omega - G)/R^* \leqq k_2$, where k_1, k_2 are positive constants depending only on the geometry of Σ. Therefore

$$b_0 = \int\limits_\beta^\alpha \left|\frac{ds}{d\varphi}\right| d\varphi \leqq (R^*(\beta)/k_1)\,(\alpha - \beta), \tag{68}$$

where $R^*(\beta) = \max\limits_{p^* \in D^*(\beta)} \overline{Op^*}$. Combining (54), (66), (67) and (68), and putting $R_0^* = \min\limits_{p^* \in \Sigma^*} \overline{Op^*}$, we have

$$|K(q)| \leqq \left[\frac{R^*(\beta)\,4K_1 n_2}{R_0^{*\,2}\,m_1 k_1 c}\right]^2 \frac{(\alpha - \beta)^2}{d^2}. \tag{69}$$

For the case in which S is a surface of the form $z = \varphi(x,y)$, the above results are valid with $\beta = \frac{1}{2}\pi$. $D(\frac{1}{2}\pi)$ is the region of the σ, τ plane bounded by the convex curve $\Gamma(\frac{1}{2}\pi)$ in which Σ^* cuts the σ, τ plane, and $D^*(\frac{1}{2}\pi)$ is the region of Σ^* for which $G(\sigma,\tau) < 0$. For all points $(\sigma,\tau) \in D(\frac{1}{2}\pi)$, let $b(\sigma,\tau)$ denote the distance along the ray through the origin from (σ,τ) to the boundary of $D(\frac{1}{2}\pi)$. Let $b_0 = b(\sigma(0,0), \tau(0,0))$ and $G_0 = G(\sigma(0,0), \tau(0,0))$. Let $B = \max\limits_{(\sigma,\tau) \in D(\frac{1}{2}\pi)} (-b/G)$.

Clearly B is a finite positive number, since $(-b/G)$ is a positive continuous function in the interior of $D(\frac{1}{2}\pi)$ and takes the values of $1/G_\Omega$ on the boundary. Thus

$$b_0 \leqq B G_0. \tag{70}$$

The image under the normal mapping of a point $(x, y, \varphi(x, y))$ on S is the point $(\sigma, \tau, G(\sigma, \tau))$ on Σ^*; hence for all $(\sigma, \tau) \in D(\frac{1}{2}\pi)$

$$W = (1 + \varphi_x^2 + \varphi_y^2)^{\frac{1}{2}} = \left(1 + (\sigma/G)^2 + (\tau/G)^2\right)^{\frac{1}{2}} = -(R^*/G). \tag{71}$$

Now, let $(x_0, y_0, \varphi(x_0, y_0))$ be the coordinates of the point q on S. Let $W_0 = \left(1 + \varphi_x^2(x_0, y_0) + \varphi_y^2(x_0, y_0)\right)^{\frac{1}{2}}$, and let r be the radius of the largest circle about (x_0, y_0) in which $\varphi(x, y)$ is defined. Clearly $r < d$; hence, combining (54), (66), (67), (70) and (71), we have

$$K(x_0, y_0) \leqq \left[\frac{4 K_1 n_2 B R^*(\frac{1}{2}\pi)}{m_1 c R_0^*}\right]^2 \frac{1}{W_0^2 r^2}. \tag{72}$$

In order to prove the inequality (52), we shall need the following well known lemma.

Lemma 11. *Let S be the surface defined by $z = \varphi(x, y)$ where $\varphi(x, y)$ is a solution of an equation of minimal surface type; i.e., an equation of the form $a(u, v)\varphi_{xx} + 2b(u, v)\varphi_{xy} + c(u, v)\varphi_{yy} = 0$, where the coefficients satisfy the inequality* (29). *The conclusion is that* (a) *the spherical mapping is quasi-conformal with eccentricity $\leqq e_1$, where e_1 is a constant which depends only on the equation; and* (b) *the ratio of the magnitudes of the principal curvatures of S is bounded above and below by the positive constants $(\delta_1)^{\frac{1}{2}}$, $(\delta_1)^{-\frac{1}{2}}$ respectively, where $\delta_1 = e_1 + (e_1^2 - 1)^{\frac{1}{2}}$.*

Proof. By means of the definitions of Section II, take u, v to be the local coordinates on the hemisphere U which contains the spherical image of S. The spherical mapping is then given by $u = \varphi_x$, $v = \varphi_y$. The condition of minimal surface type implies that the metric $ds^2 = a\,du^2 + 2b\,du\,dv + c\,dv^2$ is quasi-conformally related to the Euclidean metric on U, with eccentricity bounded by a constant e_0. This same condition also implies that taken as a metric on S, $\Phi(ds^2) = W^4(c\,dx^2 - 2b\,dx\,dy + a\,dy^2)$ is quasi-conformally related to the Euclidean metric on S, with eccentricity bounded by the same constant e_0. As in Sections II, III the spherical mapping is conformal in the metrics $\Phi(ds^2)$, ds^2. The spherical mapping may therefore be obtained by composition from a conformal mapping and two mappings of eccentricity $\leqq e_0$. The eccentricity of such a mapping is readily shown to be bounded by $e_1 = 2e_0^2 - 1$. Part (b) of Lemma 11 is valid for any surface whose spherical mapping is quasi-conformal with eccentricity $\leqq e_1$. A proof follows easily from the classical formula of differential geometry

$$III - 2H\,II + K\,I = 0, \tag{73}$$

where III, II, I are the first, second and third fundamental forms, and H, K are the mean curvature and the Gaussian curvature respectively. The normal curvature k in a direction making angle φ with a principal direction on S is given by

$$k = II/I = k_1 \sin^2\varphi + k_2 \cos^2\varphi, \tag{74}$$

where k_1 and k_2 are the principal curvatures. Using (73) and putting $\delta = (\max III/I)/(\min III/I)$, we see that the eccentricity E of the spherical mapping is given by

$$E = \tfrac{1}{2}(\delta + 1/\delta) = \tfrac{1}{2}\left((k_1/k_2)^2 + (k_2/k_1)^2\right), \tag{75}$$

and the result follows by requiring that $E \leqq e_1$.

Now let S be a solution surface of the parametric variational problem under consideration of the form $z = \varphi(x, y)$. At a point of S, let k_1 be the principal curvature of greatest magnitude. From part (b) of Lemma 11 it follows that $|k_1| \leqq (\delta_1)^{\frac{1}{2}} |k_2|$. Thus if k is the normal curvature in any direction, then $|k| \leqq (k_1^2)^{\frac{1}{2}} \leqq (\delta_1 |K|)^{\frac{1}{2}}$. Now let $k_{(x)}$, $k_{(y)}$ denote the normal curvature in the directions $dy = 0$, $dx = 0$ respectively; then

$$k_{(x)} = \frac{\varphi_{xx}}{W(1+\varphi_x^2)}, \qquad k_{(y)} = \frac{\varphi_{yy}}{W(1+\varphi_y^2)}, \tag{76}$$

and it follows that

$$\begin{aligned}(1/W^4)\left[(\varphi_{xx}+\varphi_{yy})^2 + 2(\varphi_{xy}^2 - \varphi_{xx}\varphi_{yy})\right] &\leqq W^2\left(|k_{(x)}| + |k_{(y)}|\right)^2 + 2|K| \\ &\leqq (4W^2\delta_1 + 2)|K|.\end{aligned} \tag{77}$$

The inequality (52) now follows by estimating $K(x_0, y_0)$ by means of (51).

VII. In this section we consider the problem of determining all 1-forms on Σ whose Φ-images are exact differentials on solution surfaces. We have noted in Section V that a 1-form on Σ will have this property if and only if in each coordinate patch U its coefficients with respect to the local coordinates u, v satisfy the system (33). As previously mentioned, such a 1-form is a generalization of a conservation law. It is clear that in order to determine an exact differential on a particular solution surface S, a 1-form which is a generalized conservation law need be defined only in a sub-region of Σ containing the image of S under the normal mapping. We now show that for regular variational problems, the generalized conservation laws can be characterized globally in terms of solutions of a single uniformly elliptic system. Let S be a solution surface whose normal directions omit the positive z direction. Let $d\psi = \Phi(h)$ be an exact differential on S which arises from a 1-form h on Σ. Let Σ^* be the surface $\omega = G(\sigma, \tau)$ constructed according to Lemma 6. As before, let p_0 be the point in which the positive ω-axis intersects Σ. As noted above, in this case we need only consider h in the sub-region $\Sigma - p_0$. The projection through the origin of $\Sigma - p_0$ onto Σ^*, followed by the projection parallel to the ω-axis onto the σ, τ plane, defines a $1-1$ mapping of $\Sigma - p_0$ onto the σ, τ plane. We may therefore take σ, τ to be local coordinates in $\Sigma - p_0$. In terms of σ, τ h is given by an expression $h = A(\sigma, \tau)\,d\sigma + B(\sigma, \tau)\,d\tau$. If $\bar{V}$ is a coordinate patch of S corresponding to a coordinate patch $\bar{U}$ of Σ^*, then in terms of the local coordinates in $\bar{V}$, $\bar{U}$

$$h = (A\sigma_{\bar{u}} + B\tau_{\bar{u}})\,d\bar{u} + (A\sigma_{\bar{v}} + B\tau_{\bar{v}})\,d\bar{v}, \tag{78}$$

and

$$d\psi = \Phi(h) = \bar{F}^2\left((A\sigma_{\bar{v}} + B\tau_{\bar{v}})\,d\bar{x} - (A\sigma_{\bar{u}} + B\tau_{\bar{u}})\,d\bar{y}\right). \tag{79}$$

Let ξ, η be the functions on S defined as in the proof of Theorem 3. As before, ξ, η may be taken to be independent variables, at least locally, on S; $z_\xi = \sigma$, $z_\eta = \tau$; and $z(\xi, \eta)$ is a solution of the non-parametric variational problem $\delta \int G(\sigma, \tau)\, d\xi\, d\eta = 0$. Let $\psi(x, y)$ be the function arising by integration from $d\psi$. It then follows by a calculation that $\psi_\xi = B$, $\psi_\eta = -A$, and hence $A_\xi + B_\eta = 0$. The pair (A, B) is therefore a conservation law of the variational problem $\delta \int G(\sigma, \tau)\, d\xi\, d\eta = 0$, and hence is a solution of the system

$$G_{\tau\tau} A_\sigma = G_{\sigma\sigma} B_\tau, \qquad 2G_{\sigma\tau} A_\tau = G_{\sigma\sigma}(A_\tau + B_\sigma). \tag{80}$$

Since the Euler equation of the above non-parametric variational problem is uniformly elliptic (Lemma 6), it follows that the system (80) is also uniformly elliptic. The above argument can be reversed to show that if (A, B) is a solution of the system (80), then $\Phi(A\, d\sigma + B\, d\tau)$ is an exact differential on S, wherever it is defined.

Let S be a solution surface whose normals make an angle $\geqq \beta$ with the positive z-axis, and let (A, B) be a solution of (80) in the bounded convex domain $D(\beta)$, where $D(\beta)$ is defined as in the proof of Theorem 3. The image of S on Σ^* under the normal mapping is then the region $D^*(\beta)$ which lies over $D(\beta)$, and it follows that $\Phi(A\, d\sigma + B\, d\tau)$ is defined and exact on all of S. Similarly, if S is a solution surface whose normal directions omit only the positive z direction, and (A, B) is an entire solution of (80), then $\Phi(A\, d\sigma + B\, d\tau)$ is defined and exact on all of S. Now let S be a solution surface whose normals omit no directions. Let Σ^* be as defined above. Let σ', τ', ω' be the coordinate system obtained by the rotation $\sigma' = \sigma$, $\tau' = -\tau$, $\omega' = -\omega$ from the σ, τ, ω system in the space Y. Let Σ' be the surface constructed in the primed coordinate system, according to Lemma 6, in the same fashion in which Σ^* is constructed in the unprimed system. Σ' can then be represented in the form $\omega' = G'(\sigma', \tau')$, where $G'(\sigma', \tau')$ is defined for all σ', τ'. Let p_0' be the point where the positive ω'-axis intersects Σ. As before we take σ, τ to be local coordinates in $\Sigma - p_0$, and in a similar manner we take σ', τ' to be local coordinates in $\Sigma - p_0'$. The induced mapping $\sigma(\sigma', \tau')$, $\tau(\sigma', \tau')$ is then a $1-1$ mapping of the punctured σ', τ' plane onto the punctured σ, τ plane, which carries the origin in one plane to the point at infinity in the other. Let h be a 1-form on Σ such that $\Phi(h)$ is exact on S. In terms of the local coordinates σ, τ in $\Sigma - p_0$, h is given by an expression $h = A(\sigma, \tau)\, d\sigma + B(\sigma, \tau)\, d\tau$, where (A, B) is an entire solution of the system (80). Similarly, in $\Sigma' - p_0'$, h is given by an expression $h = A'(\sigma', \tau')\, d\sigma' + B'(\sigma', \tau')\, d\tau'$, where (A', B') is an entire solution of the system

$$G'_{\tau'\tau'} A'_{\sigma'} = G'_{\sigma'\sigma'} B'_{\tau'}; \qquad 2G'_{\sigma'\tau'} A'_{\sigma'} = G'_{\sigma'\sigma'}(A'_{\tau'} + B'_{\sigma'}). \tag{81}$$

In neighborhoods of S whose normal image on Σ contains neither p_0, or p_0', we must have

$$\Phi(h) = \Phi(A'\, d\sigma' + B'\, d\tau') = \Phi(A\, d\sigma + B\, d\tau), \tag{82}$$

from which it follows that

$$A' = (G'/G)^2 (A\sigma_{\sigma'} + B\tau_{\sigma'}); \qquad B' = (G'/G)^2 (A\sigma_{\tau'} + B\tau_{\tau'}). \tag{83}$$

The requirement that (A', B') be a solution of the system (81) in a neighborhood of the origin thus imposes a condition on the growth of the functions A, B at infinity.

In the minimal surface case, Σ is the unit sphere, $G(\sigma, \tau) = \frac{1}{2}(\sigma^2 + \tau^2 - 1)$, $G'(\sigma', \tau') = \frac{1}{2}(\sigma'^2 + \tau'^2 - 1)$, and the systems (80) and (81) reduce to the Cauchy-Riemann equations. $A + iB$, $A' + iB'$ are therefore entire analytic functions of $\varrho = \sigma + i\tau$, $\varrho' = \sigma' + i\tau'$ respectively. The mappings of $\Sigma - p_0$, $\Sigma - p_0'$ onto the σ, τ plane and the σ', τ' plane respectively, are obtained by stereographic projection through the north and south poles, respectively. The relation between ϱ and ϱ' is easily seen to be simply that $\varrho = 1/\varrho'$. If we put $f(p) = A + iB$ and $f'(\varrho') = A' + iB'$, then the equations (83) become

$$f'(\varrho') = |\varrho'|^4 f(\varrho(\varrho'))\,\bar{\varrho}_{\varrho'} = -\varrho'^2 f(1/\varrho'), \tag{84}$$

where $\bar{\varrho}_{\varrho'}$ is the complex conjugate of $\varrho_{\varrho'}$. Since $f'(\varrho')$ is assumed to be regular at the origin, the entire function $f(\varrho)$ must be a polynomial of degree $\leqq 2$. We have proved the following theorem.

Theorem 4. *Let $\delta \int \mathscr{F}(\mathbf{p})\, ds\, dt = 0$, be a regular parametric variational problem. Let Σ^* be the surface $\omega = G(\sigma, \tau)$ as defined in the proof of Theorem 3. The Φ-mapping of the differential forms on Σ^* into those on a solution surface S is then well defined. Let $(A(\sigma, \tau), B(\sigma, \tau))$ be a solution of the uniformly elliptic system* (80). *Then* (a) *if the domain of (A, B) includes the bounded convex domain $D(\beta)$, the differential form $\Phi(A\, d\sigma + B\, d\tau)$ is defined and exact on every solution surface whose normal vectors make an angle $\geqq \beta$, with the positive z-axis;* (b) *if the domain of (A, B) is the entire σ, τ plane, $\Phi(A\, d\sigma + B\, d\tau)$ is defined and exact on every solution surface whose normal directions omit the positive z direction; and* (c) *if the domain of (A, B) is the entire σ, τ plane, and (A, B) satisfies the growth condition* (83) *at infinity, $\Phi(A\, d\sigma + B\, d\tau)$ can be extended to an exact differential on any solution surface, i.e., in particular on solution surfaces whose normals omit no directions. All exact differentials on solution surfaces which arise locally from conservation laws of the local non-parametric variational problems can be obtained in this way. In the minimal surface case, the system* (80) *reduces to the Cauchy-Riemann equations, and $A + iB$ is an analytic function of $\sigma + i\tau$. The growth condition in part* (c) *implies in this case that the entire function $A + iB$ is a polynomial of degree $\leqq 2$.*

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Some Properties of a Harmonic Function of Three Variables given by its Series Development

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Introduction

Integral operators transform analytic functions of one or several complex variables into solutions of linear partial differential equations. These operators formalize a procedure which enables us to derive various properties of the generated functions from corresponding results for analytic functions of complex variables (see [*6*]). In the present paper we shall consider the operator

$$H(\boldsymbol{X}) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} f_1(u,\zeta) \frac{d\zeta}{\zeta}, \qquad \boldsymbol{X} = (x, y, z), \tag{1}$$

which transforms an analytic function $f_1(u,\zeta)$ of $u = x + \frac{i y}{2}(\zeta + \zeta^{-1}) + \frac{1}{2} z(\zeta - \zeta^{-1})$ and ζ into a harmonic function of three real variables. The generated function $H(\boldsymbol{X})$ is complex-valued; taking its real part, we obtain a real harmonic function (see [*1, 2, 3, 4, 5, 6, 9, 10, 13, 15, 17*]). We assume at first that f_1 is of the form

$$f_1(u,\zeta) = f(u)\,\zeta^{\varkappa} \tag{2}$$

where f is a meromorphic function of one complex variable and we investigate relations between the coefficients of the function element of $H(\boldsymbol{X})$ at the origin and the location of singularities and growth of $H(\boldsymbol{X})$. When we attempt to develop the analogues of the classical theorems, various complications arise which have to be overcome.

In the present paper the following questions are considered:

I. The operator (1) generates complex functions, while we are primarily interested in corresponding relations for real harmonic functions, $Q(\boldsymbol{X})$.

In §2 we indicate how the coefficients change when we pass from the development of a real harmonic function $Q(\boldsymbol{X}) = \mathrm{Re}[H(\boldsymbol{X})]$ to that of a complex harmonic function $H(\boldsymbol{X})$ and *vice versa.*

II. The right-hand side of (1) represents the functions $H(\boldsymbol{X})$ only in *the domain of association of the integral operator* ([*6*] p. 49). By an analytic continuation $H(\boldsymbol{X})$ is defined "in the large"* (*i.e.*, in the whole domain of its existence). When

* As indicated in [*4*] the analytic continuation of the generated function can be obtained by varying the integration curve $\mathscr{L}$ in the operator $\frac{1}{2\pi i}\int\limits_{\mathscr{L}} f(u,\zeta)\,\frac{d\zeta}{\zeta}$.

formulating the results for the harmonic function, we have to determine the region where they are valid.

III. The associate function $f(u)$ is defined in the plane, while $H(\boldsymbol{X})$ is a function of three variables. In the case when the associate (2) is rational the generated function $H(\boldsymbol{X})$ is algebraic in x, y, z; its singularities, corresponding to the poles of $f(u)$, lie along circles and segments of straight lines. In some cases these circles degenerate to singular points. In order to formulate conveniently various relations between the coefficients of the development of the real harmonic function, we introduce certain matrices, (1.3)* and (1.14) below. The coefficients of the development (1.1) of a (complex) harmonic function form a triangular matrix (1.13), which is a sum of one-column matrices (1.14) of the development

$$
\begin{gathered}
H(\boldsymbol{X}) = \sum_{n=0}^{\infty} a_n \Gamma_{NM}(\boldsymbol{X}) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} a_n \int\limits_{|\zeta|=1} (u^m \zeta^p)^n \zeta^{\varkappa-1}\, d\zeta, \\
N = m\,n, \quad N = p\,n + \varkappa, \quad m, p, \varkappa \text{ are fixed (real) constants.}
\end{gathered} \tag{3}
$$

Using results of HADAMARD [11] and NEVANLINNA [16], in § 1 we derive connections between the coefficients of the development (3) and the density branch lines of the function $H(\boldsymbol{X})$. See [5], p. 549. In § 2 we determine how the coefficients of the development of a real harmonic function vary if we rotate the coordinate system. This yields a generalization of the results of § 1. In § 3 a theorem referring to the structure of certain essential singularities is proved. In § 4 bounds for harmonic functions with circular branch lines in terms of the coefficients are derived. In § 5 we indicate some generalizations of our results.

§ 1. Relations between the coefficients of the development of a harmonic function and the density of its branch lines

A (complex) harmonic function of three variables regular at the origin can be developed in a sufficiently small neighborhood $\mathscr{V}(O)$ of the origin O in a series of the form** (see [6] (21), p. 42)

$$
\sum_{n=0}^{\infty} \sum_{\varkappa=0}^{2n} a_{n\varkappa} \Gamma_{n\varkappa}(\boldsymbol{X}) = \sum_{n=0}^{\infty} \sum_{\varkappa=0}^{2n} a_{n\varkappa} \frac{n!\, i^{|\varkappa-n|}}{(n+|\varkappa-n|)!} r^n P_{n,|n-\varkappa|}(\cos\vartheta)\, e^{i(\varkappa-n)\varphi}, \quad \boldsymbol{X} \in \mathscr{V}(O), \tag{1}
$$

where

$$
\Gamma_{n\varkappa}(\boldsymbol{X}) \overset{\text{def}}{=} \frac{1}{2\pi i} \int\limits_{|\zeta|=1} u^n \zeta^{-n+\varkappa-1}\, d\zeta \tag{1a}
$$

are complex spherical harmonics multiplied by constants; r, ϑ and φ are spherical coordinates. When generalizing results of the theory of analytic functions of one complex variable*** to the theory of harmonic functions, the following difficulty arises: while for functions of one variable we usually consider the

* (1.3) – (3) of § 1.

** We note that in formula (21) of [6] the summation $\sum\limits_{n=0}^{\infty} \sum\limits_{m=-n}^{n}$ is used.

*** In particular, this is the case when we consider theorems about the connections between a series development and the location of singularities.

series $\sum_{n=0}^{\infty} a_n u^n$, in the case of three-dimensional harmonic functions the development (1) in spherical harmonics is no longer a simple series. In this connection it is useful to consider at first special developments in spherical harmonics:

$$S(\boldsymbol{X}) = \sum_{n=0}^{\infty} a_n \Gamma_{N,K}(\boldsymbol{X}), \tag{2}$$

where $N = m n$, $K = (m+p) n + \varkappa$; $m, p, \varkappa$ are fixed integers and $0 \leqq K \leqq 2N$ for $n \geqq n_0$, n_0 sufficiently large.

We investigate here the simplest case of series of the form (2), namely,

$$S(\boldsymbol{X}) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \sum_{n=0}^{\infty} a_n u^n \zeta^{\varkappa-1} \, d\zeta. \tag{3}$$

Let $f(u)$ be a meromorphic function regular at the origin O, where f has the development $f(u) = \sum_{n=0}^{\infty} a_n u^n$, let α be a complex number, $\alpha \neq a_0$. We shall consider the harmonic function

$$G_\alpha(\boldsymbol{X}) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{\zeta^{\varkappa-1}\, d\zeta}{(f(u)-\alpha)}. \tag{4}$$

Let $A_1(\alpha), A_2(\alpha), \ldots, |A_\nu(\alpha)| \leqq |A_{\nu+1}(\alpha)|$ be the zeros of $[f(u)-\alpha]$. By the theorem of MITTAG-LEFFLER, we have

$$g_\alpha(u) = \frac{1}{(f(u)-\alpha)} = \sum_{\nu=1}^{\infty} \left\{ \sum_{\mu=1}^{\sigma_\nu} \frac{M_{\nu\mu}}{(u - A_\nu(\alpha))^\mu} - p_\nu(u) \right\} + e(u) = \sum_{\nu=1}^{\infty} s_\nu(u) + e(u), \tag{5}$$

where σ_ν are positive integers, M_ν are complex numbers, $p_\nu(u)$ are polynomials and $e(u)$ is an entire function. The series $\sum_{\nu=1}^{\infty} s_\nu(u)$ converges uniformly in every closed bounded set which does not contain the poles $A_\nu(\alpha)$ of $g_\alpha(u)$. Therefore, we can interchange the order of summation and integration in

$$G_\alpha(\boldsymbol{X}) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \left[\sum_{\nu=1}^{\infty} s_\nu(u) + e(u) \right] \zeta^{\varkappa-1} \, d\zeta. \tag{6}$$

Whence

$$\begin{aligned} G_\alpha(\boldsymbol{X}) = \sum_{\nu=1}^{\infty} \Bigg\{ & \frac{M_{\nu 0}}{\sqrt{(x - A_\nu(\alpha))^2 + y^2 + z^2}} \left[\frac{-(x - A_\nu(\alpha)) + [(x - A_\nu(\alpha))^2 + y^2 + z^2]^{\frac{1}{2}}}{i y + z} \right]^\varkappa + \\ & + \cdots + (-1)^{\sigma_\nu} M_{\nu \sigma_\nu} \frac{\partial^{\sigma_\nu - 1}}{\partial x^{\sigma_\nu - 1}} \Bigg[\frac{1}{((x - A_\nu(\alpha))^2 + y^2 + z^2)^{\frac{1}{2}}} \times \\ & \times \left[\frac{-(x - A_\nu(\alpha)) + [(x - A_\nu(\alpha))^2 + y^2 + z^2]^{\frac{1}{2}}}{i y + z} \right]^\varkappa \Bigg] - P_\nu(\boldsymbol{X}) \Bigg\} + E(\boldsymbol{X}) \\ & \stackrel{\text{def}}{=} \sum_{\nu=1}^{\infty} S_\nu(\boldsymbol{X}) + E(\boldsymbol{X}), \end{aligned} \tag{7}$$

where $P_\nu(\boldsymbol{X})$ is a harmonic polynomial of $\boldsymbol{X} = (x, y, z)$ and $E(\boldsymbol{X})$ is an entire harmonic function.

We shall show that the series $\sum_{\nu=1}^{\infty} S_\nu(\boldsymbol{X})$ converges uniformly in every subdomain of $x^2+y^2+z^2<\infty$ which does not contain the singularities of $S_\nu(\boldsymbol{X})$, $\nu=1, 2, \ldots$. Let $\mathscr{K}_R=(x, y, z\,|\,x^2+y^2+z^2<R^2)$. For every $R<\infty$, $R\neq|A_\nu(\alpha)|$, we divide the sum $\sum_{\nu=1}^{\infty}\cdots$ into two parts

$$\sum_{\nu=1}^{N}\cdots+\sum_{\nu=N+1}^{\infty}\cdots$$

so that $|A_\nu(\alpha)|<R$ for $\nu\leqq N$ and $|A_\nu(\alpha)|>R$ for $\nu>N$. If we apply the operator $\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\cdots\frac{d\zeta}{\zeta}$ to the second sum, we obtain a harmonic function which is regular in $\mathscr{K}_R$, since $|x+iy\cos t+iz\sin t|^2\leqq x^2+y^2+z^2\leqq R^2$ for $(x, y, z)\in\mathscr{K}_R$, while $|A_\nu(\alpha)|>R$, for $\nu>N$.

From (7) we deduce that $S_\nu(\boldsymbol{X})$ has a branch line of the second order along the circle

$$x=\operatorname{Re}A_\nu(\alpha), \qquad y^2+z^2=\big(\operatorname{Im}A_\nu(\alpha)\big)^2, \tag{8}$$

if $\operatorname{Im}A_\nu(\alpha)\neq 0$. If $\operatorname{Im}A_\nu(\alpha)=0$, then the branch line shrinks to a point. We call this point a degenerate branch line*.

If $\operatorname{Im}A_\nu=0$ and $\varkappa=0$ [see (4)], then $S_\nu(\boldsymbol{X})$ is single-valued and infinite at the point $x=A_\nu(\alpha)$, $y=z=0$. If $\varkappa\neq 0$, then the function $S_\nu(\boldsymbol{X})$ becomes infinite along the segment $[A_\nu\leqq x\leqq\infty,\ y=z=0]$ if $A_\nu>0$, and along $[-\infty\leqq x\leqq A_\nu,\ y=z=0]$ if $A_\nu<0$. See [*6*], p. 45ff.

Thus we see that the branch lines and the degenerate branch lines of $G_\alpha(\boldsymbol{X})$ lie on the spheres

$$x^2+y^2+z^2=|A_\nu(\alpha)|^2, \qquad \nu=1, 2, \ldots. \tag{9}$$

We observe that the poles $A_\nu(\alpha)$ of the function $g_\alpha(u)=\frac{1}{f(u)-\alpha}$ are the α-points of $f(u)$. Applying a theorem of NEVANLINNA ([*16*], p. 72), we obtain

Lemma 1.1. *Let*

$$G_\alpha(\boldsymbol{X})=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\zeta^{\varkappa-1}\,d\zeta}{[f(u)-\alpha]}, \tag{10}$$

where $f(u)$ is a meromorphic function of u, $f(0)\neq\alpha$, and let $A_\nu(\alpha)$ be the zeros of $[f(u)-\alpha]$. Then

(1^0) *the harmonic function $G_\alpha(\boldsymbol{X})$ has branch lines*** *which lie on the spheres*

$$x^2+y^2+z^2=|A_\nu(\alpha)|^2. \tag{11}$$

(2^0) *If the series*

$$\sum_{\nu=1}^{\infty}\frac{1}{|A_\nu(\alpha)|^\lambda}, \qquad \lambda<\infty, \tag{12}$$

converges for three different values of α, say $\alpha=\alpha_\mu$, $\mu=1, 2, 3$, then it must converge for all α, $\alpha\neq f(0)$.

---

* If the function $S_\nu(\boldsymbol{X})$ is continued to complex values of x, y, z, it has a branch manifold $\mathscr{B}$ and $A_\nu(\alpha)\in\mathscr{B}$.

** If $\operatorname{Im}A_\nu(\alpha)=0$, the corresponding branch line degenerates to a point.

In order to be able to formulate some of the results without referring to the associate functions, we shall connect the study of singularities with properties of coefficients $a_{n\varkappa}$ of the development (1) of $H(\boldsymbol{X})$. The coefficients $\{a_{n\varkappa}\}$ form a triangular matrix

(13)
$$[a_{n\varkappa}] = \begin{pmatrix} & & & a_{00} & & & \\ & & a_{10} & a_{11} & a_{12} & & \\ & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

which can be written as a sum of one-column matrices

(14)
$$\begin{pmatrix} a_{00} \\ a_{11} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} + \begin{pmatrix} 0 \\ a_{10} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} + \begin{pmatrix} 0 \\ a_{12} \\ a_{23} \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ a_{20} \\ a_{31} \\ \cdot \\ \cdot \end{pmatrix} + \cdots$$
$$= [a_{N,M};\, N=n,\, M=n] + [a_{N,M};\, N=n-1,\, M=n-2] + + [a_{N,M};\, N=n-1,\, M=n] + \cdots$$

where $a_{N,M}=0$ if $N<0$ or $M<0$. In order that $\sum_{n=0}^{\infty} a_n u^n$ be a function element of a meromorphic function for $|u|<\infty$, the necessary and sufficient condition is that

(15)
$$\lim_{j\to\infty} \frac{l_j}{l_{j-1}} = 0, \qquad l_p = \overline{\lim_{n\to\infty}} |D_n^{(p)}|^{\frac{1}{p}},$$

$$D_n^{(p)} = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+p} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+p+1} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n+p} & a_{n+p-1} & \cdots & a_{n+2p} \end{vmatrix} = (-1)^{\frac{p(p+1)}{2}} \begin{vmatrix} a_{n+p} & \cdots & a_n \\ a_{n+p+1} & \cdots & a_{n+1} \\ \cdot & \cdots & \cdot \\ a_{n+2p} & \cdots & a_{n+p} \end{vmatrix}$$

$$= (-1)^{\frac{p(p+1)}{2}} ((a_{n+p+\vec{\nu}+\overset{\downarrow}{\mu}}))_{\substack{\nu=0,-1,\ldots,-p,\\ \mu=0,1,\ldots,p}}$$

where (( )) is a Toeplitz matrix. (See [*8*], p. 335, [*11*].)

Theorem 1.1. *Let the constants* $\{a_n\}$ *of the development* (2), $N=n$, $K=n$, *satisfy the condition* (15)*, *and let*

(16)
$$F(\boldsymbol{X}) = \sum_{n=0}^{\infty} a_n \Gamma_{n,n}(\boldsymbol{X})$$

be one-column series. Further, let

(17)
$$G_\alpha(\boldsymbol{X}) + E(\boldsymbol{X}) = \sum_{n=0}^{\infty} b_n(\alpha)\, \Gamma_{N_1,M_1}(\boldsymbol{X}) + E(\boldsymbol{X}), \qquad \boldsymbol{X}\in\mathscr{V}(O),$$

$\star$ $\sum_{n=0}^{\infty} a_n u^n$ is the function element at the origin of a meromorphic function.

where $N_1 = n - c_1$, $M_1 = n - c_2$. *Here* c_1 *and* c_2 *are constants (integers) and*

$$b_0(\alpha) = \frac{1}{(a_0 - \alpha)}$$

(18)
$$b_M(\alpha) = \frac{(-1)^M}{(a_0-\alpha)^{M+1}} \Big[\det\big((a_{i-j+1})\big)_{\substack{j=1,2,\ldots,M-1\\ k=1,2,\ldots,M-1\\ a_k=0 \text{ for } k<0}} - \alpha \det E_1\Big]$$

$$= \frac{(-1)^M}{(a_0-\alpha)^{M+1}} \begin{vmatrix} a_1 & a_0-\alpha & 0 & \ldots & 0 \\ a_2 & a_1 & a_0-\alpha & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ a_{M-1} & a_{M-2} & a_{M-3} & \ldots & a_0-\alpha \\ a_M & a_{M-1} & a_{M-2} & \ldots & a_1 \end{vmatrix}.$$

Here (( )) *is a Toeplitz matrix,*

$$E_1 = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

and $E(\boldsymbol{X})$ *is an entire harmonic function.*

Then the only branch lines (or degenerate branch lines) of the second order of the function $G_\alpha(\boldsymbol{X})$ *are circles. These branch lines lie on the spheres* $[x^2 + y^2 + z^2 = |A_\nu(\alpha)|^2]$, $\nu = 1, 2, \ldots$. *The radii* $|A_\nu(\alpha)|$, $\nu = 1, 2, \ldots$, *have the property indicated under* 2^0, *p. 210* (see Lemma 1).

Theorem 1 can be generalized in various directions. Instead of assuming that we have one column, $G_\alpha(\boldsymbol{X})$, we can consider a sum of finitely many such one-column series.

Further, one can use the associate functions

(19)
$$\sum_{n=0}^{\infty} a_n (u^N \zeta^P)^n \zeta^\varkappa,$$

where N, P, $\varkappa$ are fixed constants and $|P| < N$. In this case, instead of circles, we obtain, in general, certain algebraic curves as singular lines.

§ 2. Harmonic functions with circular branch lines in the parallel planes $\alpha_1 x + \alpha_2 y + \alpha_3 z = \beta_\nu$, $\nu = 1, 2, 3, \ldots$

In § 1 *complex* harmonic functions have been considered. One can easily formulate these results in terms of the real part of the development (1.1) and (1.2). We have

$$\frac{1}{i^{|\varkappa-n|}} \Gamma_{n\varkappa}(\boldsymbol{X}) = \frac{1}{2\pi i^{|\varkappa-n|+1}} \int_{|\zeta|=1} u^n \zeta^{-n+\varkappa} \frac{d\zeta}{\zeta}$$

(1)
$$= \frac{n!}{(n+|\varkappa-n|)!} r^n P_{n,|\varkappa-n|}(\cos\vartheta)\,[\cos(\varkappa-n)\varphi + i\sin(-\varkappa+n)\varphi]$$

$$\overset{\text{def}}{=} Q_{n,2|\varkappa-n|}(\boldsymbol{X}) + i \operatorname{sgn}(\varkappa-n)\, Q_{n,2|\varkappa-n|-1}(\boldsymbol{X}), \qquad \operatorname{sgn} 0 = 0.$$

Let $q_{n\mu}$, $\mu=0, 1, \ldots, 2n$, $n=0, 1, 2, \ldots$ be real numbers. Then

$$\begin{aligned}
&\sum_{\mu=0}^{2n} q_{n\mu}\, Q_{n\mu}(\boldsymbol{X}) = \operatorname{Re}\Big\{q_{n0}\, Q_{n0}(\boldsymbol{X}) + \\
(2)\qquad &+\sum_{\varkappa=1}^{n} \left(q_{n,2|\varkappa|} - i \operatorname{sgn}(\varkappa-n)\, q_{n,2|\varkappa|-1}\right) \left(Q_{n,2|\varkappa|}(\boldsymbol{X}) + i \operatorname{sgn}(\varkappa-n)\, Q_{n,2|\varkappa|-1}(\boldsymbol{X})\right)\Big\} \\
&= \operatorname{Re}\Big\{\sum_{\varkappa=-n}^{n} a_{n,n-\varkappa}\, \Gamma_{n,n-\varkappa}(\boldsymbol{X})\Big\},
\end{aligned}$$

where

$$(3)\qquad \tfrac{1}{2} a_{n,n-\varkappa}\, i^{|\varkappa-n|} = q_{n,2|n|} - i \operatorname{sgn}(-\varkappa)\, q_{n,2|n|-1}, \qquad x \neq 0.$$

(3) shows how the coefficients change if one passes from the series in *complex* harmonic functions $\Gamma_{n\varkappa}(\boldsymbol{X})$ to the development in real (modified) spherical harmonics $Q_{n\varkappa}(\boldsymbol{X})$ and *vice versa*.

The branch lines in Lemma 1.1 lie in the parallel planes $x=c_\nu$, where c_ν are constants. We shall derive conditions on the coefficients $q_{n\varkappa}$ in order that the branch lines lie in finitely many sets of parallel planes

$$\overset{(\varrho)}{\alpha}_{11}\, x + \overset{(\varrho)}{\alpha}_{21}\, y + \overset{(\varrho)}{\alpha}_{31}\, z = \overset{(\varrho)}{c}_{\nu}, \qquad \nu = 1, 2, 3, \ldots; \quad \varrho = 1, 2, \ldots, \mathsf{P}; \quad \mathsf{P} < \infty.$$

Every rotation of three-dimensional space,

$$(4)\qquad \begin{aligned}
x &= \alpha_{11}^{(1)} x^{(1)} + \alpha_{12}^{(1)} y^{(1)} + \alpha_{13}^{(1)} z^{(1)}, \\
y &= \alpha_{21}^{(1)} x^{(1)} + \alpha_{22}^{(1)} y^{(1)} + \alpha_{23}^{(1)} z^{(1)}, \\
z &= \alpha_{31}^{(1)} x^{(1)} + \alpha_{32}^{(1)} y^{(1)} + \alpha_{33}^{(1)} z^{(1)}, \\
&\sum_{\nu=1}^{3} \alpha_{m\nu}^{(1)} \alpha_{n\nu}^{(1)} = \delta_{mn}
\end{aligned}$$

(see [*14*], p. 103), transforms a real homogeneous polynomial into a homogeneous polynomial of the same degree. Therefore,

$$(5)\qquad Q_{n\varkappa}(\boldsymbol{X}) = \sum_{\mu=0}^{2n} d_{n\mu}^{(\varkappa)}\, Q_{n\mu}(\boldsymbol{X}^{(1)}),$$

where $d_{n\mu}^{(\varkappa)}$ are real constants (see (2)). Let

$$(6)\qquad \boldsymbol{d}_{n\mu} = (d_{n\mu}^{(0)}, d_{n\mu}^{(1)}, \ldots, d_{n\mu}^{(2n)})$$

and

$$(7)\qquad \boldsymbol{q}_n = (q_{n0}, q_{n1}, \ldots, q_{n,2n})$$

be vectors in $(2n+1)$-dimensional space. Then

$$(8)\qquad \sum_{\varkappa=0}^{2n} q_{n\varkappa} \sum_{\mu=0}^{2n} d_{n\mu}^{(\varkappa)}\, Q_{n\mu}(\boldsymbol{X}^{(1)}) = \sum_{\mu=0}^{2n} (\boldsymbol{q}_n \cdot \boldsymbol{d}_{n\mu})\, Q_{n\mu}(\boldsymbol{X}^{(1)}),$$

where $(\boldsymbol{q}_n \cdot \boldsymbol{d}_{n\mu}) = \sum_{\varkappa=0}^{2n} q_{n\varkappa}\, d_{n\mu}^{(\varkappa)}$ is the inner product of the vectors $\boldsymbol{q}_n$ and $\boldsymbol{d}_{n\mu}$. Let

$$(9)\qquad h(\boldsymbol{X}) = \sum_{n=0}^{\infty} \sum_{\varkappa=0}^{2n} q_{n\varkappa}\, Q_{n\varkappa}(\boldsymbol{X}), \qquad \boldsymbol{X} \in \mathscr{V}(0),$$

be the development of a real harmonic function $h(\boldsymbol{X})$, regular at the origin (see [19], p. 393). If we rotate the coordinate system (see (4)), for the transformed function we obtain the development

$$h(\boldsymbol{X}) = h^{(1)}(\boldsymbol{X}^{(1)}) = \sum_{n=0}^{\infty} \sum_{\mu=0}^{2n} (\boldsymbol{q}_n \cdot \boldsymbol{d}_{n\mu})\, Q_{n\mu}(\boldsymbol{X}^{(1)}), \qquad \boldsymbol{X}^{(1)} \in \mathscr{V}(O). \tag{10}$$

By (2) and (3),

$$h(\boldsymbol{X}) = h^{(1)}(\boldsymbol{X}^{(1)}) = \mathrm{Re}\left\{\sum_{h=0}^{\infty} \sum_{\varkappa=-n}^{n} a_{n,n-\varkappa}^{(1)} \Gamma_{n,n-\varkappa}(\boldsymbol{X}^{(1)})\right\}, \tag{11}$$

where

$$\tfrac{1}{2} a_{n,n-\varkappa}^{(1)} = (\boldsymbol{q}_n \cdot \boldsymbol{d}_{n,2|k|}) - i \operatorname{sgn}(-\varkappa)\, (\boldsymbol{q}_n \cdot \boldsymbol{d}_{n,2|\varkappa|-1}). \tag{12}$$

Thus we see that

$$\begin{aligned} h^{(1)}(\boldsymbol{X}^{(1)}) = \mathrm{Re}\Bigg\{\sum_{n=0}^{\infty} \sum_{\varkappa=-n}^{n} \Bigg[a_{n,n-\varkappa}^{(1)} \frac{i^{|n-\varkappa|}\, n!}{(n+|n-\varkappa|)!} \times \\ \times r_1^n P_{n,|n-\varkappa|}(\cos\vartheta_1) \big(\cos(\varkappa-n)\varphi_1 + i \sin(\varkappa-n)\varphi_1\big)\Bigg]\Bigg\}. \end{aligned} \tag{13}$$

Here $r_1 = r, \vartheta_1, \varphi_1$ are polar coordinates in $x^{(1)}, y^{(1)}, z^{(1)}$-space. We shall show that the series in brackets $\{\ldots\}$ converges uniformly and absolutely in a sufficiently small neighborhood $\mathscr{V}(O)$ of the origin. $h^{(1)}(\boldsymbol{X}^{(1)})$ is a real harmonic function regular in $\mathscr{V}(O)$. According to [18] and [19], p. 399, $h^{(1)}(\boldsymbol{X}^{(1)})$ can be developed in a uniformly and absolutely convergent series

$$\begin{aligned} h^{(1)}(\boldsymbol{X}^{(1)}) = \sum_{n=0}^{\infty} \Bigg(\sum_{\varkappa=-n}^{n} \frac{\mathrm{Re}(i^{|n-\varkappa|} a_{n,n-\varkappa})\, n!}{(n+|n-\varkappa|)!} r_1^n P_{n,|n-\varkappa|}(\cos\vartheta_1) \cos(\varkappa-n)\varphi_1 - \\ - \sum_{\varkappa=-n}^{n} \frac{\mathrm{Im}(i^{|n-\varkappa|} a_{n,n-\varkappa})\, n!}{(n+|n-\varkappa|)!} r_1^n P_{n,|n-\varkappa|}(\cos\vartheta_1) \sin(\varkappa-n)\varphi_1 \Bigg). \end{aligned} \tag{14}$$

If we set $\varphi_1 = 0$, we see that

$$\sum_{n=0}^{\infty} \sum_{\varkappa=-n}^{n} \left[\frac{\mathrm{Re}(a_{n,n-\varkappa}^{(1)} i^{|n-\varkappa|})\, n!}{(n+|n-\varkappa|)!} r_1^n P_{n,|n-\varkappa|}(\cos\vartheta_1)\right] \tag{15}$$

converges uniformly and absolutely in $\mathscr{V}(O)$. If we differentiate (14) with respect to φ_1 and then set $\varphi_1 = 0$, we see that the series

$$\sum_{n=0}^{\infty} \sum_{\varkappa=-n}^{n} \left[\frac{\mathrm{Im}(a_{n,n-\varkappa}^{(1)} i^{|n-\varkappa|})\, n!}{(n+|n-\varkappa|)!} r_1^n P_{n,|n-\varkappa|}(\cos\vartheta_1)\right] \tag{16}$$

converges uniformly in $\mathscr{V}(O)$.

In this way we obtain the complex harmonic function

$$\begin{aligned} H^{(1)}(\boldsymbol{X}^{(1)}) &= \sum_{n=0}^{\infty} \sum_{\varkappa=-n}^{n} a_{n\varkappa}^{(1)} \Gamma_{n\varkappa}^{(1)}(\boldsymbol{X}^{(1)}), \qquad \boldsymbol{X}^{(1)} \in \mathscr{V}(O), \\ h^{(1)}(\boldsymbol{X}^{(1)}) &= \mathrm{Re}[H^{(1)}(\boldsymbol{X}^{(1)})], \end{aligned} \tag{17}$$

where the coefficients $a_{n,n-\varkappa}^{(1)}$ are given by (12).

Corollary. *Let $q_{n\varkappa}$ be the coefficients of the development* (9) *of a real harmonic function which is regular at the origin, and let every $q_{n\varkappa}$ be a sum of s numbers,*

$$q_{n\varkappa} = \sum_{\varrho=1}^{s} \overset{(\varrho)}{q}_{n\varkappa}. \tag{18}$$

The $\overset{(\varrho)}{q}_{n\varkappa}$ possess the following property: Let

$$\left(\overset{(\varrho,\varkappa_\varrho)}{l_\mu} \Big/ \overset{(\varrho,\varkappa_\varrho)}{l_{\mu-1}}\right) \to 0 \quad \text{for} \quad \mu \to \infty \tag{19}$$

where

$$\overset{(\varrho,\varkappa_\varrho)}{l_\mu} = \overline{\lim_{n\to\infty}} \left|\overset{(\varrho,\varkappa_\varrho)}{D_{n\mu}}\right|^{\frac{1}{n}}, \tag{20}$$

$$\overset{(\varrho,\varkappa_\varrho)}{D_{n\mu}} = \begin{vmatrix} \overset{(\varrho)}{a}_{n,n+\varkappa_\varrho} & \overset{(\varrho)}{a}_{n+1,n+1+\varkappa_\varrho} & \cdots & \overset{(\varrho)}{a}_{n+\mu,n+\mu+\varkappa_\varrho} \\ \overset{(\varrho)}{a}_{n+1,n+1+\varkappa_\varrho} & \overset{(\varrho)}{a}_{n+2,n+2+\varkappa_\varrho} & \cdots & \overset{(\varrho)}{a}_{n+\mu+1,n+\mu+1+\varkappa_\varrho} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \overset{(\varrho)}{a}_{n+\mu,n+\mu+\varkappa_\varrho} & \overset{(\varrho)}{a}_{n+\mu+1,n+\mu+1+\varkappa_\varrho} & \cdots & \overset{(\varrho)}{a}_{n+2\mu,n+2\mu+\varkappa_\varrho} \end{vmatrix}. \tag{21}$$

The coefficients $\overset{(\varrho)}{a}_{n,n-\varkappa}$ and $\overset{(\varrho)}{q}_{n,\varkappa}$ are connected by the relation (12). *Then*

$$h(\boldsymbol{X}) = \sum_{\varrho=1}^{s} h_\varrho(\boldsymbol{X}); \tag{22}$$

here the harmonic function

$$h_\varrho(\boldsymbol{X}) = \sum_{n=0}^{\infty} \sum_{\varkappa=0}^{2n} \overset{(\varrho)}{q}_{n\varkappa} Q_{n\varkappa}(\boldsymbol{X}), \qquad \boldsymbol{X} \in \mathscr{V}(O), \qquad \varrho = 1, 2, \ldots, s, \tag{23}$$

has pole-like singularities with branch lines (degenerate branch lines) lying in the planes

$$\overset{(\varrho)}{\alpha}_{11} x + \overset{(\varrho)}{\alpha}_{21} y + \overset{(\varrho)}{\alpha}_{31} z = \text{const.}, \qquad \varrho = 1, 2, \ldots, s. \tag{24}$$

Therefore, Lemma 1.1 and Theorem 1.1 can be formulated in terms of *real* harmonic functions.

When considering harmonic functions of *two* variables, one can introduce a composition which corresponds to multiplication of analytic functions. More precisely, if $H_k(\xi,\eta) = \operatorname{Re} f_k(\zeta)$, $\zeta = \xi + i\eta$, $k = 1, 2$, one defines the composition $*$ by

$$H_1(\xi,\eta) * H_2(\xi,\eta) = \operatorname{Re}[f_1(\zeta) f_2(\zeta)]. \tag{25}$$

An analogous composition can be defined for harmonic functions of three variables.

In (1.1a) the harmonic polynomials $\Gamma_{n\varkappa}(\boldsymbol{X})$, $\boldsymbol{X} = (x, y, z)$, were introduced. Every real harmonic function $H(\boldsymbol{X})$ regular at the origin can be written in the form

$$H(\boldsymbol{X}) = \operatorname{Re}\left\{\sum_{n=0}^{\infty} \sum_{\varkappa=0}^{2n} A_{n\varkappa} \Gamma_{n\varkappa}(\boldsymbol{X})\right\}, \qquad \boldsymbol{X} \in \mathscr{V}(O), \tag{26}$$

where $A_{n\varkappa}$ are constants, and $\mathscr{V}(O)$ is a sufficiently small neighborhood of the origin. We define the composition $*$ of harmonic polynomials of *three* variables by the relation

$$A^{(1)}_{n_1\varkappa_1}\Gamma_{n_1\varkappa_1}(\boldsymbol{X}) * A^{(2)}_{n_2\varkappa_2}\Gamma_{n_2\varkappa_2}(\boldsymbol{X}) = A^{(1)}_{n_1\varkappa_1}A^{(2)}_{n_2\varkappa_2}\Gamma_{n_1+n_2,\,\varkappa_1+\varkappa_2}(\boldsymbol{X}). \tag{27}$$

The right-hand side of (26) converges uniformly and absolutely; the convergence of the series for $H_1(\boldsymbol{X}) * H_2(\boldsymbol{X})$ follows from the convergence of developments for $H_k(\boldsymbol{X})$, $k=1, 2$. Consequently, (27) yields a composition for harmonic functions regular at the origin. See [*1*] p. 647.

The composition $*$ is associative, commutative and distributive.

Remark. The same composition can be defined by (29); see below.

If

$$H_k(\boldsymbol{X}) = \mathrm{Re}\left[\frac{1}{2\pi i}\int_{|\zeta|=1} f_k(u,\zeta)\,\frac{d\zeta}{\zeta}\right], \qquad k=1,2,3, \tag{28}$$

$$H_3(\boldsymbol{X}) = H_1(\boldsymbol{X}) * H_2(\boldsymbol{X}) = \mathrm{Re}\left[\frac{1}{2\pi i}\int_{|\zeta|=1} f_1(u,\zeta)\,f_2(u,\zeta)\,\frac{d\zeta}{\zeta}\right]. \tag{29}$$

The above result follows immediately from the representation (6), p. 40 of [*6*], of the $\Gamma_{n\varkappa}(\boldsymbol{X})$.

$H_3(\boldsymbol{X})$ will be called a q-product (quasi-product) of $H_1(\boldsymbol{X})$ and $H_2(\boldsymbol{X})$. If $H_3(\boldsymbol{X})=1$, $H_2(\boldsymbol{X})$ will be called a q-reciprocal of $H_1(\boldsymbol{X})$.

Lemma 1.1 can be reformulated as follows: *Let $G_\alpha(\boldsymbol{X})$ (see (10)), be a q-reciprocal of $[F(\boldsymbol{X})-\alpha]$. Then the functions $G_\alpha(\boldsymbol{X})$ have the properties listed in* 1^0 *and* 2^0, p. 210.

§ 3. A lower bound for the radius of an essential singular circle

To obtain further properties of singularities of a one-column harmonic function

$$S(\boldsymbol{X}) = \sum_{n=0}^{\infty} a_n \Gamma_{n,\,n+\varkappa}(\boldsymbol{X}), \qquad \boldsymbol{X}\in\mathscr{V}(O) \tag{1}$$

in terms of the a_n, we consider new combinations of the coefficients a_n. Let the associate

$$s(u) = \sum_{n=0}^{\infty} a_n u^n \zeta^\varkappa, \qquad |u|\leqq r_0, \quad r_0 \text{ sufficiently small,} \tag{2}$$

of $S(\boldsymbol{X})$ be a meromorphic function in a circle of radius R, $R<\infty$, and $s(u)$ has infinitely many poles in the neighborhood of $|u|=R$. According to the theorem of HADAMARD,

$$\lim\left(\frac{l_\mu}{l_{\mu-1}}\right) = \frac{1}{R}. \tag{3}$$

Here the l_μ are the quantities introduced in (1.15). The harmonic function $S(\boldsymbol{X})$ has an essential singularity along a circle (or at a point) on the intersection of a plane $x=$ constant and the sphere

$$x^2+y^2+z^2=R^2. \tag{4}$$

In the following we shall give a lower bound for the radius of this circle in terms of the a_n.

If $A_\nu = A_{\nu 1} + i A_{\nu 2}$, $\nu = 1, 2, \ldots$, are the poles of $s_\nu(u)$, the function $S(\boldsymbol{X}) = \sum S_\nu(\boldsymbol{X}) + E(\boldsymbol{X})$, see (1.7), has infinitely many branch lines (or degenerate branch lines) along

$$[x = A_{\nu 1},\, y^2 + z^2 = A_{\nu 2}^2]. \tag{5}$$

Two possibilities arise:

1. The $A_{\nu 2}$ go to zero for $\nu \to \infty$; in this case $S_\nu(\boldsymbol{X})$ has an essential singular point.

2. The $A_{\nu 2}$ for $\nu \to \infty$ remain larger than a positive constant; in this case $S_\nu(\boldsymbol{X})$ has an essential singular circle.

Let

$$u = c_1 w + c_2 w^2 + c_3 w^3 + \cdots \tag{6}$$

be the development of the function which maps the unit circle onto the region $\mathscr{R}$. Here $\mathscr{R}$ is the domain bounded by the segments of the straight lines $\xi = \varrho$ and $\xi = -\varrho$, $u = \xi + i\eta$ and by two arcs of the circle $|u| = R$; each of these arcs intersects the real axis. The end-points of the arcs are

$$\xi = \pm (R^2 - \varrho^2)^{\frac{1}{2}}, \qquad \eta \pm \varrho. \tag{7}$$

Let $\boldsymbol{C}_n$, $n = 0, 1, 2, \ldots$, denote a sequence of vectors

$$\boldsymbol{C}_0 = (1), \quad \boldsymbol{C}_1 = (c_1), \quad \boldsymbol{C}_2 = (c_2, c_1^2), \quad \boldsymbol{C}_3 = (c_3, 2c_1, c_2, c_1^3), \ldots. \tag{8}$$

Further let $\boldsymbol{A}_n$, $n = 0, 1, 2, \ldots$, be another sequence of vectors

$$\boldsymbol{A}_0 = (a_0), \quad \boldsymbol{A}_1 = (a_1), \quad \boldsymbol{A}_2 = (a_1, a_2), \ldots, \quad \boldsymbol{A}_n = (a_1, a_2, \ldots, a_n). \tag{9}$$

Theorem 3.1. *Suppose that the one-column development* (1) *of the harmonic function* $S(\boldsymbol{X})$ *is such that the limit* $\left|\frac{l_\mu}{l_{\mu-1}}\right| = \frac{1}{R}$, *and therefore* $S(\boldsymbol{X})$, *has an essential singularity lying on the sphere* (4). *If the quantities* $\tilde{l}_\mu$ *formed from* $\tilde{a}_n = (\boldsymbol{A}_n \cdot \boldsymbol{C}_n)$ *possess the property that*

$$\lim (\tilde{l}_\mu / \tilde{l}_{\mu-1}) = 1 \quad \textit{for} \quad \mu \to \infty, \tag{10}$$

then the essential singularity is a circle of radius $A_2 \geqq \varrho$.

Proof. By the transformation (5) the rectangle $\mathscr{R}$ (in the u-plane) is transformed into the circle $|w| \leqq 1$. The series $s(u)$ (the associate of $S(\boldsymbol{X})$) is transformed into the series

$$s\big(u(w)\big) = \sum_{n=0}^{\infty} (\boldsymbol{A}_n \cdot \boldsymbol{C}_n)\, w^n. \tag{11}$$

If the function $s(u)$ is meromorphic in $\mathscr{R}$, the function $s\big(u(w)\big)$ must be meromorphic in $|w| < 1$. If the imaginary parts $A_{\nu 2}$ of the coordinates of the poles of $s(u)$ are equal to or larger than ϱ, and a limit of a subsequence of the $A_{\nu 2}$, $\nu = 1, 2, \ldots$, equals ϱ, then

$$\lim_{\mu \to \infty} (\tilde{l}_\mu / \tilde{l}_{\mu-1}) = 1$$

where $\tilde{l}_\mu$ are the quantities introduced in (1.15) with $\tilde{a}_n = (\boldsymbol{A}_n \cdot \boldsymbol{C}_n)$.

§ 4. An upper bound for the growth of a harmonic function with a meromorphic associate

The theory of meromorphic functions of one variable yields bounds for the absolute values of analytic functions. Results of this kind can be exploited in the theory of harmonic functions of three variables.

Suppose that

$$s(u) = \frac{g(u)}{f(u)} \zeta^{\varkappa} \tag{1}$$

is the associate of a one-column harmonic function $S(\boldsymbol{X})$ (see (1.3) and (2)), where $g(u)$ and $f(u)$ are entire functions of order $\leqq \varrho$.

Let $A_\nu = A_{\nu 1} + i A_{\nu 2}$ be the zeros of $f(u)$. They are ordered in such a way that $|A_\nu| \leqq |A_{\nu+1}|$. If $|A_\nu| = |A_{\nu+\mu}|$, then $\operatorname{Arg} A_\nu < \operatorname{Arg} A_{\nu+\mu}$.

$$S(\boldsymbol{X}) = \frac{1}{2\pi i} \int_{|\zeta|=1} s(u) \zeta^{\varkappa-1} d\zeta \tag{2}$$

is defined on a Riemann manifold $\mathscr{M}$ with branch lines* of the second order along

$$x = A_{\nu 1}, \qquad y^2 + z^2 = A_{\nu 2}, \qquad \nu = 1, 2, \ldots. \tag{3}$$

As is shown in § 1, $s(u)$ can be written in the form (1.5) (with $g_\varkappa(u)$ replaced by $s(u)$), and $S(\boldsymbol{X})$ can be obtained by interchanging the order of summation and integration in

$$S(\boldsymbol{X}) = \frac{1}{2\pi i} \int_{|\zeta|=1} \left[\sum_{\nu=1}^{\infty} \sum_{\mu=1}^{\infty} s_\nu(u) + e(u) \right] \zeta^{\varkappa-1} d\zeta. \tag{4}$$

According to (7) of [6], p. 47,

$$\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\zeta^{\varkappa-1} d\zeta}{u - A_\nu} = \frac{1}{[(x-A_\nu)^2 + y^2 + z^2]^{\frac{1}{2}}} \left[\frac{-(x - A_\nu^{(\mu)})^2 + ((x-A_\nu)^2 + y^2 + z^2)^{\frac{1}{2}}}{i y + z} \right]^{\varkappa}. \tag{5}$$

We choose for $A_{\nu 1} > 0$ the branch of (5) such that in the neighborhood of $x = y = z = 0$

$$[(x - A_\nu)^2 + y^2 + z^2]^{\frac{1}{2}} = (x - A_\nu) \left[1 + \frac{1}{2} \frac{y^2 + z^2}{(x - A_\nu)^2} + \cdots \right], \tag{6}$$

and for $A_{\nu 1} < 0$ the branch

$$[(x - A_\nu)^2 + y^2 + z^2]^{\frac{1}{2}} = -(x - A_\nu) \left[1 + \frac{1}{2} \frac{y^2 + z^2}{(x - A_\nu)^2} + \cdots \right]. \tag{7}$$

(See footnote below.) We assume that $A_\nu \neq 0$, $\nu = 1, 2, \ldots$. Then $S(\boldsymbol{X})$ is regular at the origin in the first sheet of the Riemann manifold $\mathscr{M}$. The domain of association of the representation (4) is the first sheet $\mathscr{F}_1$ of $\mathscr{M}$; $\mathscr{F}_1$ is the whole space less the segments

$$x = A_{\nu 1}, \qquad y^2 + z^2 \geqq A_{\nu 2}, \qquad \nu = 1, 2, \ldots. \tag{8}$$

---

* If $A_{\nu 2} = 0$, the branch line shrinks to a point and the function is single-valued in the neighborhood of $(A_{\nu 1}, 0, 0)$.

(Those A_ν for which $A_{\nu 2}=0$ have to be omitted.) Let $\mathscr{S}=\mathscr{S}_1+\mathscr{S}_2$ be the part of the sheet $\mathscr{F}_1$ of $\mathscr{M}$ where

$$\text{(9a)}\quad \mathscr{S}_1=\sum_{\nu=1}^{\infty}[A_{\nu 1}+|A_\nu|^{-p}\leqq x\leqq A_{\nu+1,1}-|A_{\nu+1}|^{-p},\ y^2+z^2<\infty],$$

$$\text{(9b)}\quad \mathscr{S}_2=\sum_{\nu=1}^{\infty}[A_{\nu 1}-|A_\nu|^{-p}\leqq x\leqq A_{\nu 1}+|A_{\nu 1}|^{-p},\ y^2+z^2<|A_\nu|^{-2p}-(x-A_\nu)^2].$$

Here p is an arbitrary number, $\varrho<p<\infty$, where ϱ is the maximum of the orders of the functions $g(u)$ and $f(u)$.

Theorem 4.1. *Let $S(\boldsymbol{X})$ be a harmonic function with an associate $\frac{g(u)}{f(u)}\zeta^k$, where $g(u)$ and $f(u)$ are entire functions of order ϱ. Then in the subdomain $\mathscr{S}$ of the domain of association of* (4) *and for sufficiently large r the inequality*

$$\text{(10)}\qquad |S(\boldsymbol{X})|\leqq C_1 e^{r^{\varrho+\varepsilon}},\qquad C_1=C_1(\varepsilon),$$

holds. C_1 and ε are constants, $\varepsilon>0$, arbitrarily small.

Proof. We derive upper bounds for $g(u)$ and $\frac{1}{f(u)}$ separately.

1. Since $g(u)$ is an entire function of order $\leqq\varrho$, in the rectangle $|\xi|\leqq r$, $|\eta|\leqq r$ we have

$$\text{(11)}\qquad \log|g(\xi+i\eta)|\leqq\frac{k+1}{k-1}m(kr)\leqq\frac{k+1}{k-1}k^\varrho r^\varrho,\qquad k\geqq 2^{\frac{1}{\varrho}}.$$

In accordance with a theorem in the theory of entire functions ([7], p. 268), since $f(0)\neq 0$,

$$\text{(12)}\qquad |f(u)|>|f(0)|\,r\,e^{-r^{\varrho+\varepsilon}}$$

if $|u|<\infty$ and $|u-A_\nu|\geqq|A_\nu|^{-p}$, $\nu=1,2,\ldots$, $p>\varrho$. If t varies in the interval $[0\leqq t\leqq 2\pi]$, $x_0+i(y_0^2+z_0^2)^{\frac12}\cos(\varphi+t)$ varies along the straight line

$$\text{(13)}\qquad [x=x_0,\ -(y_0^2+z_0^2)^{\frac12}\leqq(y^2+z^2)^{\frac12}\leqq(y_0^2+z_0^2)]^{\frac12}.$$

If the point $(x_0,y_0,z_0)\in\mathscr{S}$, then the arc (12) lies in $\mathscr{S}$, and

$$\text{(14)}\qquad |x_0+i(y_0^2+z_0^2)^{\frac12}\cos(\varphi+t)|\leqq|x_0+i(y_0^2+z_0^2)^{\frac12}|.$$

Therefore, for $r=(x^2+y^2+z^2)^{\frac12}$ sufficiently large,

$$\text{(15)}\quad |S(\boldsymbol{X})|\leqq\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{g(x+i(y^2+z^2)^{\frac12}\cos(\varphi+t))}{f(x+i(y^2+z^2)^{\frac12}\cos(\varphi+t))}e^{it\varkappa}\right|dt\leqq\frac{e^{\left(\frac{k+1}{k-1}\right)k^\varrho r^\varrho}}{f(0)\,e^{-r^{\varrho+\varepsilon}}}\leqq C_1\,e^{r^{(\varrho+\varepsilon')}}$$

where $\varepsilon'=\frac{\log 2}{\log r}+\varepsilon$ can be made arbitrarily small for sufficiently large r.

§ 5. Generalizations

The consideration of §§ 1—4 can be generalized in various directions.

I. Let $H(\boldsymbol{X})$ be a harmonic function given by

$$\text{(1)}\quad \sum_{n=0}^{\infty}\sum_{k=0}^{n}a_n\binom{n}{k}\Gamma_{np,n(p+m)\quad k(n-1)+\varkappa}(\boldsymbol{X})=\frac{1}{2\pi i}\int_{|\zeta|=1}\sum_{n=0}^{\infty}a_n[u^p(\zeta^m+1)]^n\zeta^{\varkappa-1}d\zeta.$$

Here p, m and $\varkappa$ are integers, $p>0$, $m>0$. If the coefficients a_n satisfy the conditions

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n-1}}\right|^{1/n}=\frac{1}{|A|},\qquad \overline{\lim_{n\to\infty}}\left\|\begin{matrix}a_{n-1} & a_n\\ a_n & a_{n+1}\end{matrix}\right\|^{1/n}<\frac{1}{|A|^2}$$

(see [7], p. 323 ff.), then $H(\boldsymbol{X})$ is regular in the sphere $x^2+y^2+z^2<|A|^2$ and has the singularity given by

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=0}\frac{\zeta^{\varkappa-1}\,d\zeta}{u^p(\zeta^m+1)-A}. \tag{2}$$

Our previous considerations can be generalized to this and to analogous cases. In this way we obtain branch lines different from those considered previously.

II. Instead of $\Delta_3 G=0$, $\Delta_3=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$, we consider the equation

$$\Delta_3\Phi+A(r^2)\,\boldsymbol{X}\cdot\nabla\Phi+C(r^2)\,\Phi=0,\qquad \Phi\equiv\Phi(X), \tag{3}$$

where $A(r^2)$ and $C(r^2)$ are entire functions of r^2. As has been shown previously ([6], p. 68), the operator

$$\begin{gathered}\boldsymbol{p}_3(G(\boldsymbol{X}))\overset{\text{def}}{=}G(\boldsymbol{X})+\int_0^1 B(r^2,\sigma^2)\,G(\sigma^2 X)\,d\sigma=\Phi(\boldsymbol{X}),\qquad \boldsymbol{X}\in\mathscr{A}(\boldsymbol{p}_3,\Phi),\\ B(r^2,\sigma^2)=\sigma^2 B^*(r^2,\sigma^2)=\sigma^2\sum_{n-1}^{\infty}B^{(n)}(r^2)\,(1-\sigma^2)^{n-1},\end{gathered} \tag{4}$$

transforms harmonic functions $G(\boldsymbol{X})$ of x, y, z into solutions $\Phi(\boldsymbol{X})$ of (3). The domain $\mathscr{A}(\boldsymbol{p}_3,\Phi)$ of validity of representation (1) is called the domain of association of the representation (4). The function $B^*(r^2,\sigma^2)$ is an entire function of r^2 for $0\leqq\sigma\leqq 1$. It has the property that

$$B^*(0,\sigma^2)=0. \tag{5}$$

See [6], (10), (11), p. 68; (3), p. 64 and (11), (13), p. 65. We can continue $\Phi(\boldsymbol{X})$ and $G(\boldsymbol{X})$ to complex values of x, y, z and introduce the variables

$$X=x,\qquad Z=\frac{iy+z}{2},\qquad Z^*=\frac{iy-z}{2}. \tag{6}$$

We write $\tilde{G}(X,Z,Z^*)=G(x,y,z)$, $\tilde{\Phi}(X,Z,Z^*)=\Phi(x,y,z)$. From (5) it follows that

$$\tilde{\Phi}(2(ZZ^*)^{\frac12},Z,Z^*)=\tilde{G}(2(ZZ^*)^{\frac12},Z,Z^*), \tag{7}$$

i.e., that $\tilde{\Phi}$ and $\tilde{G}$ coincide in the branch $X=2(ZZ^*)^{\frac12}$ of the characteristic space. Instead of (modified) special harmonics used for the Laplace equation, we introduce the solutions

$$\Phi_{n,\varkappa}(\boldsymbol{X})\overset{\text{def}}{=}\boldsymbol{p}_3(\Gamma_{n\varkappa}(\boldsymbol{X})),\qquad n=0,1,2,\ldots,\quad \varkappa=0,1,\ldots,2n. \tag{8}$$

In a sufficiently small sphere (with the center at the origin) which lies in the domain of association $\mathscr{A}(\boldsymbol{p}_3,\Phi)$, $\Phi(\boldsymbol{X})$ can be represented in the form (4), since $G(\boldsymbol{X})$ can be developed in series of spherical harmonics. Therefore,

$$\Phi(\boldsymbol{X})=\sum_{n=0}^{\infty}\sum_{\varkappa=0}^{2n}a_{n\varkappa}\,\Phi_{n\varkappa}(\boldsymbol{X}),\qquad \boldsymbol{X}\in\mathscr{V}(O). \tag{9}$$

The series converges uniformly and absolutely in the sufficiently small neighborhood $\mathscr{V}(O)$ of the origin. It follows from the representation (4) that in

$\mathscr{A}(\boldsymbol{p}_3, \Phi)$ the function $\Phi(\boldsymbol{X})$ is regular in every simply connected domain which contains the origin and in which $G(\boldsymbol{X})$ is regular. The locations of the singularities of $\Phi(\boldsymbol{X})$ and $G(\boldsymbol{X})$ are closely connected. In particular, if $G(\boldsymbol{X})$ has a branch line, in many instances $\Phi(\boldsymbol{X})$ must be singular along the same line. This shows that many of the theorems about the connection between the coefficients of the development (1.1) and the location of the singularities of the harmonic function $G(\boldsymbol{X})$ yield theorems about the connections between the coefficients $a_{n\varkappa}$ of (9) and singularities of $\Phi(\boldsymbol{X})$.

As an example of a singularity of (3) which one can obtain using the integral operator $\boldsymbol{p}_3$, we consider the solution $\Phi(\boldsymbol{X})$

$$\boldsymbol{p}_3\left([(x-A\,i)^2+y^2+z^2]^{-\frac{1}{2}}\right) = [(x-A\,i)^2+y^2+z^2]^{-\frac{1}{2}} + \int_0^1 B^*(r^2,\sigma^2)\,\frac{\sigma^2\,d\sigma}{[(x\,\sigma^2-A\,i)^2+y^2+z^2]^{\frac{1}{2}}},\quad A>0. \tag{10}$$

If we approach

$$\mathfrak{s}=[x=0,\ y^2+z^2=A^2] \tag{11}$$

along $x=0$, $y^2+z^2<A^2$, the function $[(x-A\,i)^2+y^2+z^2]^{-\frac{1}{2}}$ goes to ∞.

We shall show that the second term of the right-hand side of (10) remains uniformly bounded if we approach $\mathfrak{s}$ along the same path.

$$\int_0^1 \frac{B^*(r^2,\sigma^2)\,\sigma^2\,d\sigma}{[(x\,\sigma^2-A\,i)^2+y^2\sigma^4+z^2\sigma^4]^{\frac{1}{2}}} = \frac{1}{2}\int_0^1 \frac{B^*(r^2,\tau)\,\tau^{\frac{1}{2}}\,d\tau}{r\,[(\tau-\tau_0(x,r_1))\,(\tau-\tau_1(x,r_1))]^{\frac{1}{2}}}, \tag{12}$$

where

$$\begin{gathered}\tau_0(x,r_1)=\frac{A}{r_1}\,[i\,x-(-x^2+r_1^2)^{\frac{1}{2}}],\qquad \tau_1(x,r_1)=\frac{A}{r_1}\,[i\,x+(-x^2+r_1^2)^{\frac{1}{2}}],\\ (-x^2+r_1^2)^{\frac{1}{2}}=r\left[1-\frac{1}{2}\,\frac{x^2}{r_1^2}+\cdots\right],\qquad r_1\equiv(y^2+z^2)^{\frac{1}{2}}>0.\end{gathered} \tag{13}$$

For $x=0$, we have

$$\left|\frac{1}{2}\int_0^1 \frac{B^*(r^2,\tau)\,\tau^{\frac{1}{2}}\,d\tau}{r_1\left[\left(\tau-\frac{A}{r_1}\right)\left(\tau+\frac{A}{r_1}\right)\right]^{\frac{1}{2}}}\right| \leqq \frac{1}{2}\int_0^1 \frac{\mathrm{B}\,d\tau}{r_1\,\eta\left|\tau-\frac{A}{r_1}\right|^{\frac{1}{2}}}. \tag{14}$$

For $0\leqq\tau\leqq 1$ and $0<A-r_1<\frac{1}{2}A$, we have $\left|\tau+\frac{A}{r_1}\right|\geqq\eta^2>0$, $|B^*(r^2,\tau)|\leqq\mathrm{B}<\infty$. Since $\int_0^1\frac{d\tau}{\left|\tau-\frac{A}{r_1}\right|^{\frac{1}{2}}}<\infty$, it follows from (14) that the second term of (10) remains bounded when we approach $\mathfrak{s}$ along $x=0$, $y^2+z^2<A^2$.

In the corollary on p. 215 the assumption is made that the coefficients $q_{n\varkappa}$ of the development (2.9) are sums of s constants $\overset{(\varrho)}{q}_{n\varkappa}$ and that the $\overset{(\varrho)}{q}_{n\varkappa}$ satisfy certain limit relations. (See (2.18)—(2.21).) Suppose the series development $\sum_{n=0}^{\infty}\sum_{\mu=0}^{2n} q_{n\mu}Q_{n\mu}(\boldsymbol{X})$ is given. Then arises the problem of determining whether the $q_{n\mu}$ can be represented in the form (2.18), where the $\overset{(\varrho)}{q}_{n\mu}$ satisfy some conditions of HADAMARD'S type. The author hopes to return to this question in the future.

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A Note on Harmonic Functions in $(p+2)$ Variables

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The purpose of this note is to present two integral operators which transform functions of $(p+1)$ complex variables into harmonic functions in $(p+2)$ variables. Both of these operators are generalizations of the WHITTAKER-BERGMAN operator $B_3(f, \mathcal{L}, X^0)$, which transforms functions of two complex variables into harmonic functions in three variables [*1*], [*2*].

The first operator, an immediate extension of the Bergman operator, has the form

$$H(X) = B^1_{p+2}(F; \mathcal{L}; X^0) = \left(\frac{1}{2\pi i}\right)^p \int_{\mathcal{L}} F(\sigma; \zeta) \frac{d\zeta}{\zeta}, \star$$

$$\begin{aligned}\sigma = x_1 &+ \frac{i x_2}{2}\left(\zeta_1 + \frac{1}{\zeta_1}\right) + \frac{x_3}{4}\left(\zeta_1 - \frac{1}{\zeta_1}\right)\left(\zeta_2 + \frac{1}{\zeta_2}\right) + \cdots + \\ &+ \frac{(-i)^p}{2^p} x_{p+1}\left(\zeta_1 - \frac{1}{\zeta_1}\right)\left(\zeta_2 - \frac{1}{\zeta_2}\right)\cdots\left(\zeta_{p-1} - \frac{1}{\zeta_{p-1}}\right)\left(\zeta_p + \frac{1}{\zeta_p}\right) + \\ &+ \frac{(-i)^{p+1}}{2^{p+1}} x_{p+2}\left(\zeta_1 - \frac{1}{\zeta_1}\right)\left(\zeta_2 - \frac{1}{\zeta_2}\right)\cdots\left(\zeta_{p-1} - \frac{1}{\zeta_{p-1}}\right)\left(\zeta_p - \frac{1}{\zeta_p}\right),\end{aligned}$$

the domain of integration $\mathcal{L} = \prod_{k=1}^{p} \mathcal{L}_k$ is a product of regular contours $\mathcal{L}_k$ in the ζ_k-planes, $\|X - X^0\| < \varepsilon$, $X \equiv (x_1, x_2, \ldots, x_{p+2})$, $X^0 \equiv (x^0_1, x^0_2, \ldots, x^0_{p+2})$, and $\varepsilon > 0$ is sufficiently small.

σ has been chosen in such a manner that σ^n is automatically a hyperspherical harmonic, and consequently one has $\sum_{j=1}^{p+2} \frac{\partial^2}{\partial x_j^2} F(\sigma; \zeta) = 0$. We may realize how the operator $B^1_{p+2}(F; \mathcal{L}; X^0)$ transforms the functions $F(\sigma; \zeta)$ into harmonic functions $H(X)$ if we consider σ^n as the generating function for a class of harmonic polynomials. We introduce these homogeneous, harmonic polynomials of degree n, $h_n(X; m)$, by the relation$\star\star$ $\sigma^n = \sum_{m=-n}^{+n} h_n(X; m)\, \zeta^m$. With the residue theorem

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$\star$ Here and in future instances we shall use the contracted notations $F(\sigma; \zeta)$ and $\frac{d\zeta}{\zeta}$ to represent $F(\sigma; \zeta_1, \zeta_2, \ldots, \zeta_p)$ and $\frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} \cdots \frac{d\zeta_p}{d\zeta_p}$, respectively.

$\star\star$ Here we have introduced the notation $\sum_{m=-n}^{+n} h_n(X; m)\, \zeta^m$, which is to mean the p-tuple sum $\sum_{m_1=-n}^{+n} \sum_{m_2=-n}^{+n} \cdots \sum_{m_p=-n}^{+n} h_n(X; m_k)\, \zeta_1^{m_1} \zeta_2^{m_2} \ldots \zeta_p^{m_p}$. This abreviation will be used throughout the remainder of this work.

we may then show that the functions of $(p+1)$ complex variables

$$F(\sigma;\zeta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} a_n(m)\, \sigma^n \zeta^m$$

map onto the harmonic functions

$$H(X) \equiv B^1_{p+2}(F;\mathscr{L};0) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} a_n(-m)\, h_n(X;m).$$

There is, however, a major disadvantage in using this operator, in that σ^n does not generate linearly independent polynomials*. Because of this we now turn to another integral operator, which does generate linearly independent harmonic polynomials.

We introduce the operator

$$B^2_{p+2}(F;\mathscr{L};X^0) = A \int_{\mathscr{L}} F(\tau) \left[\frac{1}{2i}\left(\zeta - \frac{1}{\zeta}\right)\right]^m \frac{d\zeta}{\zeta} \qquad (m_k = p-k-2;\ k=1,2,\ldots,p).$$

$$\tau_k = -\frac{1}{4}\,\frac{t_{k+1}}{t_k}\left(\zeta_k - \frac{1}{\zeta_k}\right)^2 \sin\vartheta_k, \qquad (k=1,2,\ldots,p-1),$$

$$\tau_0 = t_1 r, \quad \text{and} \quad \tau_p = -\frac{1}{4}\,\frac{e^{\pm i\varphi}}{t_p}\left(\zeta_p - \frac{1}{\zeta_p}\right)\sin\vartheta_p,$$

where

$$t_k = \cos\vartheta_k + \frac{i}{2}\sin\vartheta_k\left(\zeta_k + \frac{1}{\zeta_k}\right), \qquad (k=1,2,\ldots,p),$$

$$\|X - X^0\| < \varepsilon, \qquad X \equiv (x_1, x_2, \ldots, x_{p+2}), \qquad X^0 \equiv (x_1^0, x_2^0, \ldots, x_{p+2}^0),$$

$\varepsilon > 0$ is sufficiently small, and the domain of integration $\mathscr{L} = \prod_{k=1}^{p} \mathscr{L}_k$ is a product of regular contours $\mathscr{L}_k$ joining $+1$ to -1 in the ζ_k-planes. The ϑ_k, φ, and r are hyperspherical polar coordinates, and may be defined by [3]

$$\begin{aligned}
x_1 &= r\cos\vartheta_1,\\
x_2 &= r\sin\vartheta_1\cos\vartheta_2,\\
x_3 &= r\sin\vartheta_1\sin\vartheta_2\cos\vartheta_3,\\
&\ \vdots\\
x_{p+2} &= r\sin\vartheta_1\sin\vartheta_2\ldots\sin\vartheta_p\sin\varphi,
\end{aligned}$$

where $0 \leqq \vartheta_j \leqq \pi$ $(j=1,2,\ldots,p)$, and $0 \leqq \varphi \leqq 2\pi$; the constant A is defined by

$$A = \prod_{k=0}^{p-1} A_{k+1} = \prod_{k=0}^{p-1} \frac{\Gamma(m_{k+1}+m_k+p-k)}{(m_k - m_{k+1})!\left[\Gamma\left(m_{k+1}+\frac{p}{2}-\frac{k}{2}\right)\right]^2}.$$

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* There are only $(2n+p)\dfrac{(n+p-1)!}{p!\,n!}$ linearly independent harmonic polynomials of degree n [3].

This operator was developed by considering the following functions, which form a complete, linearly independent set of harmonic polynomials of degree n [3]:

$$H(m_k;\pm;X)\equiv\left(\frac{x_{p+1}}{r_p}+i\,\frac{x_{p+2}}{r_p}\right)^{\pm m_p} r_p^{m_p}\prod_{k=1}^{p} r_{k-1}^{\nu_k}\,C_{\nu_k}^{\mu_k}(x_k/r_{k-1})$$

$$\left(\mu_k=m_k+\frac{p}{2}+\frac{1}{2}-\frac{k}{2},\quad\text{and}\quad \nu_k=m_{k-1}-m_k\right),$$

$$\equiv r^n\,Y(m_k;\vartheta_k;\varphi)=r^n\,e^{\pm m_p\varphi}\prod_{k=1}^{p}(\sin\vartheta_k)^{m_k}\,C_{\nu_k}^{\mu_k}(\cos\vartheta_k),$$

where

$$r_k=(x_{k+1}^2+x_{k+2}^2+\cdots+x_{p+2}^2)^{\frac12},\qquad r=r_0,$$

$k=0,1,\ldots,p$, and $n=m_0\geqq m_1\geqq\cdots\geqq m_p\geqq 0$, and where the $C_\nu^\lambda(\cos\vartheta_k)$ are Gegenbauer polynomials. A harmonic function in $(p+2)$ variables which is regular in a neighborhood of the origin may be expanded in terms of these polynomials as a series*

$$H(X)=\sum_{n=0}^{\infty}\sum_{m=0}^{m'}\beta_n(m)\,r^n\,Y(m;\vartheta;\pm\varphi)$$

that converges within the largest hypersphere inside of which there is no singularity of $H(X)$. The operator $B_{p+2}^2(F;\mathscr{L},X^0)$ is then obtained by expressing the $H(m_k;\pm;X)$ as combinations of the integral representations**

$$C_\nu^\lambda(\cos\vartheta)=\frac{\pi^{-\frac12}\,\Gamma(\nu+2\lambda)\,\Gamma(\nu+\frac12)}{\Gamma(\lambda)\,\Gamma(2\lambda)\,\Gamma(\nu+1)}\int_0^\pi[\cos\vartheta+i\sin\vartheta\cos t]^\nu\,[(\sin t)^{2\lambda-1}\,dt.$$

An inverse operator for $B_{p+2}^2(F;\mathscr{L};X^0)$ may be obtained by making use of the orthogonality property of the $H(m_k;\pm;X)$ on the unit hypersphere [3]; that is,

$$\iint_\Omega H(m_k;\pm;X/r)\,\overline{H(n_k;\pm;X/r)}=(2\pi)\prod_{k=1}^{p}\delta_{m_k}^{n_k}\,E_k(m_{k-1},m_k),$$

where

$$E_k(m_{k-1},m_k)=\frac{\pi\,2^{k-2m_k-p}\,\Gamma(m_{k-1}+m_k+p+1-k)}{\left(m_{k-1}+\frac{p}{2}+\frac12-\frac{k}{2}\right)(m_{k-1}-m_k)!\left[\Gamma\left(m_k+\frac{p}{2}+\frac12-\frac{k}{2}\right)\right]^2}.$$

For instance, let $H(X)=B_{p+2}^2(F;\mathscr{L};X^0)$, where $F(\tau)$ is of the form

$$F(\tau)=\sum_{n=0}^{\infty}\sum_{m=0}^{m'}\alpha_n(m)\,p(m;\tau);$$

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* The expression $\sum_{m=0}^{m'}\beta_n(m)\,r^n\,Y(m;\vartheta;\pm\varphi)$ is meant to designate the p-tuple sum $\sum_{m_1=0}^{m_0}\sum_{m_2=0}^{m_1}\cdots\sum_{m_p=0}^{m_{p-1}}\beta_n(m_k)\,r^n\,Y(m_k;\vartheta;\pm\varphi)$.

** It may be seen that the $H(m_k;\pm;X)$ are generated by $B_{p+2}^2(F;\mathscr{L};X^0)$ from the terms $p(m;\tau)=\prod_{k=0}^{p}\tau_k^{m_k}$.

then there exists an inverse operator

$$B^2_{p-2}(F;\mathscr{L};X^0)^{-1} \quad C_{p-1}(H;\Omega;\tau^0) = F(\tau) = \frac{1}{2\pi^{p-1}} \iint\limits_{\Omega} H(X)\,\overline{K(\tau|\vartheta;\pm\varphi)}\,d\Omega$$

(where the integration is over the unit hypersphere), which maps the harmonic functions in $(p+2)$ variables $H(X)$ onto the analytic functions of $(p+1)$ variables $F(\tau)$. This may be seen by choosing

$$K(\tau|\vartheta;\pm\varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^{m'} e^{\pm m_p\varphi} [(\bar\tau \sin\vartheta)^m \mu\, C^{\mu}_{\nu}(\cos\vartheta)]$$

$$\left(u_k = m_k + \frac{p}{2} + \frac{1}{2} - \frac{k}{2}, \quad \nu_k = m_{k-1} - m_k, \quad \text{and} \quad k = 1, 2, \ldots, p\right),$$

applying the orthogonality properties, and noticing that

$$E_k(m_{k-1}, m_k) = \frac{\pi A_k}{\left(m_{k-1} + \frac{p}{2} + \frac{1}{2} - \frac{k}{2}\right)}.$$

We shall now state without proof a theorem concerning the singularities of harmonic functions generated by an integral operator $B^j_{p+2}(F;\mathscr{L};X^0)$, $(j=1, 2)$. The proof is essentially the same in structure to those developed by the author for three and four dimensional harmonic functions [4], [5].

Theorem 1. *If $Z^{p+2} = E\{S(X;\zeta)=0\}$ is the singularity manifold of the integral operator $B^j_{p+2}(F;\mathscr{L};X^0)$ $(j=1,2)$, then the harmonic function $H(X) = B^j_{p+2}(F;\mathscr{L};X^0)$ is regular at the point X providing this point does not lie on the intersection*

$$E\{S(X;\zeta)=0\} \cap E\left\{\frac{\partial S}{\partial\zeta_1} + \sum_{k=2}^{p} \frac{\partial S}{\partial\zeta_k}\frac{d\psi_k}{d\zeta_1} = 0\right\},$$

where the $\zeta_k = \psi_k(\zeta_1)$ are arbitrary functions of ζ_1.

We note, that certain special cases occur when the ζ_k $(k=2, 3, \ldots, p)$ are independent of ζ_1, for instance a particular class of singularities may occur when X lies simultaneously on the surfaces

$$S(X;\zeta) = 0, \qquad \frac{\partial}{\partial\zeta_i} S(X;\zeta) = 0, \qquad (i = 1, 2, \ldots, p).$$

We now turn to a few examples to illustrate the use of Theorem 1. For instance, let us consider the case where $F(\tau)$ is a function of τ_0 alone, that is $F(\tau) = f(\tau_0)$, and where $f(\tau_0)$ has a singularity at $\tau_0 = \alpha$. By making use of the identities $\cos\vartheta_k = x_k/r_{k-1}$, $\sin\vartheta_k = r_k/r_{k-1}$ $(k=1, 2, \ldots, p)$, we may represent the singularity manifold as

$$S(X;\zeta_1) \equiv (\tau_0 - \alpha)\,\zeta_1 = \zeta_1 x_1 + \frac{i r_1}{2}(\zeta_1^2 + 1) = 0.$$

Eliminating ζ_1 between $S(X;\zeta_1) = 0$ and $\frac{\partial S}{\partial\zeta_1} = 0$ yields $(x_1 - \alpha)^2 + r_1^2 = 0$, which was to be expected, since these are the singularities of the generalized axially symmetric potentials [6].

As another example, let us consider the case where $F(\tau)$ is merely a function of $\tau_j\left(\zeta_j - \frac{1}{\zeta_j}\right)^{-2}$, $(1 \leq j \leq p-1)$, and has a singularity at $\tau_j\left(\zeta_j - \frac{1}{\zeta_j}\right)^{-2} = \alpha$. Here,

he singularity manifold of the integrand may be represented by

$$S(X;\zeta_j,\zeta_{j+1}) \equiv \zeta_j\Big(\zeta_{j+1}\,x_{j+1} + \frac{i}{2}\,r_{j+1}[\zeta_{j+1}^2+1]\Big) + \\ + 4\alpha\,\zeta_{j+1}\Big(\zeta_j\,x_j + \frac{i}{2}\,r_j[\zeta_j^2+1]\Big) = 0.$$

'he only possible singularities of $H(X)$ occur then for X simultaneously on $S\big(X;\zeta_j,\pi(\zeta_j)\big)=0$, and $\frac{\partial}{\partial\zeta_j}S\big(X;\zeta_j,\pi(\zeta_j)\big)=0$, where $\zeta_{j-1}=\pi(\zeta_j)$ is an arbitrary unction of ζ_j. For the purpose of illustration let $\pi(\zeta_j)=\zeta_j$; then upon eliminating $_j$ from $S=0$, $\frac{\partial S}{\partial\zeta_j}=0$, one obtains the locus of possible singularities

$$(x_{j+1}+4\alpha\,x_j)^2+(r_{j+1}+4\alpha\,r_j)^2=0.$$

In an earlier paper [*6*], the author showed that there are certain distinguished oints associated with an operator which transforms analytic functions into eneralized axially symmetric potentials; indeed, it was shown that a branch f a potential function might have singularities on its axis of symmetry, which lo not correspond to singularities of the analytic function*. This situation may e seen to occur here also for the case where $S(X;\zeta_j,\zeta_{j+1})$ is given as above, nd we consider the class of possible singularities for which X lies on $S=0$, $\frac{\partial S}{\partial\zeta_j}=0$, nd $\frac{\partial S}{\partial\zeta_{j+1}}=0$. One then obtains $x_{j+1}=x_{j+2}=\cdots=x_{p+2}=0$ $(j=1,2,\ldots,p)$. Ve note that this is a close analogy to the axially symmetric case, and it occurs ndoubtedly for the same reason.

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* See Z. Nehari [*7*] for another operator having similar properties.

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Application of Conformal Mapping to Boundary Perturbation Problems for the Membrane Equation

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1. Introduction

LIN has suggested [*5*] that conformal transformations are useful for boundary perturbation problems involving general differential equations. It is required that the boundary conditions be given on some curve B_ε near to a curve B_0 for which the problem can be solved; the conformal mapping of B_0 onto B_ε will then differ only slightly from the identity mapping. Since we consider equations more general than Laplace's, the transformed equation will differ from the original one, but by a small amount, so that the solution can then be determined by standard perturbation methods.

In another paper [*10*], the author has shown that LIN's suggestion leads to an efficient method of solving certain problems of current interest in fluid mechanics, for example the two-dimensional time-dependent flow of a *viscous* incompressible fluid forced by a circular cylinder vibrating about a point slightly displaced from the center of a surrounding circular cylinder. In this paper we report the results of work on the eigenvalue problem

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \lambda W = 0 \quad \text{in } R_\varepsilon, \qquad W = 0 \quad \text{on } B_\varepsilon, \tag{1}$$

where R_ε is a region in the z-plane ($z = x + i y$) bounded by B_ε. This problem is the simplest one which includes all the features we wish to consider but is already sufficiently general to describe significant physical situations. For example, the discussion in [*1*] on the validity of the Born-Oppenheimer approximation requires treating (1) when B_ε is a parallelogram and B_0 is a rectangle. The work which follows, particularly § 5, would be at once applicable.

Several methods have been proposed for finding eigenvalues in boundary perturbation problems. (See [*7*], Chapter 9, and the references given there.) Although each of these is useful in appropriate instances, to the author's knowledge no convergence proof has been given, and at least one method generally soon diverges. In contrast, we give conditions guaranteeing convergence of the mapping-perturbation (MP) method for sufficiently small perturbations. For a large class of interesting cases, these conditions are expressed simply and geometrically. To take a specific example, the method definitely converges for the principle eigenvalue of an ellipse of eccentricity less than about $\frac{1}{20}$.

Often more important in practice than the convergence of a perturbation method is the degree of approximation obtained from the first few terms. For this reason we compare the MP method with three other methods for attacking (1) when B_ε is an ellipse of small eccentricity.

Finally, we show that, when B_ε is nearly an isosceles right triangle, the eigenfunction perturbation series in the original variables must have certain singularities which are absent for the series in mapped variables. It appears that perturbations of the independent variables by conformal mapping can offer analogous advantages for elliptic equations to those known to be possessed [5] by perturbations in characteristic independent variables for hyperbolic equations.

We begin by outlining the MP method for (1) and applying it to ellipses of small eccentricity.

2. Some formal calculations

Assume that an eigenvalue $\lambda^{(0)}$ and the associated eigenfunction $W^{(0)}$ are known for the region R_0 "near" the region R_ε of (1). The formal procedure for obtaining λ is then to transform the region R_0 (taken in the ζ plane, where $\zeta=\xi+i\eta$) by means of the conformal mapping $z=z(\zeta;\varepsilon)$ into the region R_ε. With this, (1) becomes

$$\frac{\partial^2 w}{\partial \xi^2}+\frac{\partial^2 w}{\partial \eta^2}+\lambda J w=0 \quad \text{in } R_0, \qquad w=0 \quad \text{on } B_0, \tag{2}$$

where $w(\xi,\eta)=W[x(\xi,\eta),y(\xi,\eta)]$. The Jacobian J is assumed to have a series expansion of the form

$$J\equiv\left|\frac{dz}{d\zeta}\right|^2\equiv 1+\varepsilon J_1(\xi,\eta)+\varepsilon^2 J_2(\xi,\eta)+\cdots, \tag{3}$$

so that we have transformed the original boundary perturbation problem (1) into a standard perturbation problem. Formulas for obtaining the series

$$\lambda=\lambda^{(0)}+\varepsilon\lambda^{(1)}+\cdots, \tag{4}$$

$$w=w^{(0)}+\varepsilon w^{(1)}+\cdots, \tag{5}$$

can now be derived by a straightforward extension of the usual perturbation calculations used on ordinary differential equations, as in [13], p. 219–223.

As an example, let us consider the region bounded by

$$z\bar z-\left(\frac{\varepsilon}{2}\right)(z^2+\bar z^2)=1, \tag{6}$$

which is an ellipse of eccentricity $\left(\frac{2\varepsilon}{1+\varepsilon}\right)^{\frac12}$. The required series (3) for $\left|\frac{dz}{d\zeta}\right|^2$ can be found from formulas in [4] expressing the mapping of $\xi^2+\eta^2<1$ onto the region bounded by the *analytic near-circle*

$$z\bar z+\varepsilon F(z,\bar z)=1, \qquad F \text{ analytic near } \quad z\bar z=1. \tag{7}$$

The mapping is given by $z=\zeta+\frac12\varepsilon\zeta^3+\varepsilon^2\left(-\frac{\zeta}{8}+\frac{\zeta^5}{2}\right)+\cdots$ so that, with $\zeta=\varrho\exp(i\varphi)$, $\left|\frac{dz}{d\zeta}\right|^2=1+3\varepsilon\varrho^2\cos 2\varphi+\cdots$.

Let λ_{mnc} and λ_{mns} denote the eigenvalues associated with the eigenfunctions $J_m(\alpha_{mn}\varrho)\cos m\varphi$ and $J_m(\alpha_{mn}\varrho)\sin m\varphi$, J_m being the m^{th}-order Bessel function with $J_m(\alpha_{mn})=0$, $n=1, 2, \ldots$. In the series

$$\lambda_{mnc}=\alpha_{mn}^2+\varepsilon\,\lambda_{mnc}^{(1)}+\cdots,\qquad \lambda_{mns}=\alpha_{mn}^2+\varepsilon\,\lambda_{mns}^{(1)}+\cdots,$$

application of the appropriate formulas to our example shows that

$$\begin{gathered}\lambda_{mnc}^{(1)}=\lambda_{mns}^{(1)}=0,\qquad m\neq 1,\\ \lambda_{1ns}=\lambda_{1nc}=3\alpha_{1n}^2 J_2^{-2}(\alpha_{1n})\int_0^1 \varrho^2 J_1^2(\alpha_{1n}\varrho)\,d\varrho=\tfrac{1}{2}\lambda_{1n}^{(0)},\end{gathered}\tag{8}$$

the integral being evaluated with the aid of [*15*], p. 138.

It is natural to compare this example of the MP method with the classical use of Mathieu functions, which involves the conformal mapping of the interior of the ellipse onto the interior of a rectangle. As in [*6*], p. 212, the k^{th} approximation to the lower eigenvalues can be found by computing a root of a polynomial of degree $2(k-1)$. Another natural attack is to employ the Joukowski transformation, $z=\zeta+c^2\zeta^{-1}$, with which it is possible to work out a perturbation procedure for ellipses of small eccentricity. One computes the first approximations to the eigenvalues by determining the zeros of an expression involving Bessel functions. Since the mapping is of the interior of an ellipse onto a circular annulus, however, these equations are somewhat complicated and are useful only for the problem of a region bounded by two confocal ellipses.

In order to compare the Mathieu, Joukowski, and MP methods, we have calculated λ_{11c} and λ_{11s} for the ellipse (6) when $\varepsilon=0.0204$; the results are given in the Table. It is seen that the MP method is an efficient one even for this special problem: it seems to converge better than the Mathieu method, although the calculations at each stage for the latter are easier. In addition, convergence of the MP series is proved below, while the convergence of the Mathieu method seems never to have been proved.

Table. *λ_{11c} and λ_{11s}*

Method	Approximation		
	1st	2nd	3rd
JOUKOWSKI	15.05	—	—
MP	14.67	14.53, 14.83	—
MATHIEU	14.39	14.53, 14.87	14.53, 14.84

It is natural to ask how the calculations would proceed under a non-conformal transformation. One such approach is to write (6) as $r=(1-\varepsilon\cos 2\vartheta)^{-\frac{1}{2}}$ and then to make the change of variable $r(1-\varepsilon\cos 2\vartheta)^{\frac{1}{2}}=\varrho$, $\vartheta=\varphi$. The differential equation becomes

$$\begin{aligned}&\left[1-\varepsilon\cos 2\varphi+\frac{\varepsilon^2\sin^2 2\varphi}{1-\varepsilon\cos 2\varphi}\right]\frac{\partial^2 w}{\partial\varrho^2}+\\ &\quad+\frac{1}{\varrho}\left[1-\varepsilon\cos 2\varphi+\frac{2\varepsilon\cos 2\varphi+\varepsilon^2(\sin^2 2\varphi-2)}{1-\varepsilon\cos 2\varphi}\right]\frac{\partial w}{\partial\varrho}+\\ &\quad+\frac{1}{\varrho^2}\left[1-\varepsilon\cos 2\varphi\right]\frac{\partial^2 w}{\partial\varphi^2}+\frac{1}{\varrho}\,2\varepsilon\sin 2\varphi\,\frac{\partial^2 w}{\partial\varrho\,\partial\varphi}+\lambda w=0.\end{aligned}\tag{9}$$

The boundary condition is $w=0$ on $\varrho=1$. Proceeding with the calculations, we again obtain the MP result (8), although the expression leading to it is different. This is not surprising since (9) contains ε analytically and should therefore lead to convergent series for the eigenvalues. The complicated nature of (9) would make a proof difficult, just as it makes calculations difficult.

3. A convergence theorem

The simplest convergence result is the following: We assume (at first) that the unperturbed eigenvalue is simple, and that B_0 is composed of a finite number of curves possessing a continuously turning tangent such that the angle at which two such curves meet (as seen from inside R_0) is non-zero. We further assume that the mapping function $z=z(\zeta, \varepsilon)$ is such that for some constants c and K, the expansion

$$\left|\frac{dz}{d\zeta}\right| = 1 + \varepsilon\,\varphi_1(\xi,\eta) + \varepsilon^2\,\varphi_2(\xi,\eta) + \cdots \tag{10}$$

converges in the mean to $\left|\frac{dz}{d\zeta}\right|$, the coefficients satisfying

$$\iint_{R_0} \varphi_n^4\, d\xi\, d\eta \leqq K^2 c^{4n}, \qquad n=1,2,\ldots. \tag{11}$$

If d is the distance from $\frac{1}{\lambda^{(0)}}$ to the reciprocal of the nearest unperturbed eigenvalue, and Γ is defined as in (18), we define ε_1 by

$$\varepsilon_1 = \left(\frac{d}{2c}\right)(d+4\Gamma K)^{-1}. \tag{12}$$

If $|\varepsilon|$ is less than ε_1, then (4) converges, (5) converges uniformly for (ξ,η) in R_0, and the terms in (4) and (5) are correctly given by the formal calculations.

As was discovered after the present work was completed, Titchmarsh recently discussed eigenvalue problems for perturbed differential equations (not for perturbed boundaries). He proved for (2) that if the function represented by J satisfies certain inequalities, then the formal perturbation calculations are valid [*13*]. Titchmarsh's results would require modification to suit the purposes for which we shall need them, but his work is sufficiently general to render inadvisable a full presentation of the proof of our theorem. Consequently, we shall limit ourselves to an outline of our argument and refer any interested reader to [*11*] for details. Our proof is an application of known theorems in the perturbation theory for linear symmetric operators in a Hilbert space. This theory was originally expounded in a series of papers by Rellich [*9*]; we use the refined version of Nagy [*12*]. (Titchmarsh's work makes no explicit use of Hilbert space theory but "is adapted ... from the analysis of Rellich".)

In order to apply the Rellich-Nagy theory, instead of considering (1), we consider the equivalent integral equation

$$W(x,y) = \lambda \iint_{R'_\varepsilon} G_\varepsilon(x,y;x',y')\, W(x',y')\, dx'\, dy'. \tag{13}$$

G_ε is the Green's function of ∇^2 vanishing on B_ε, the primes denote variables of integration, and R'_ε is the (x',y') region corresponding to the region R_ε for

(x, y) ([3], p. 365). Changing variables of integration, and providing symmetry by defining a new kernel

$$K(\xi,\eta;\xi',\eta';\varepsilon)\equiv G_0(\xi,\eta;\xi',\eta')\,[J(\xi,\eta;\varepsilon)\,J(\xi',\eta';\varepsilon)]^{\frac{1}{2}}, \tag{14}$$

we can write (13) as

$$\iint\limits_{R_0'} K(\xi,\eta;\xi',\eta';\varepsilon)\,u(\xi',\eta';\varepsilon)\,d\xi'\,d\eta' = \Lambda\,u(\xi,\eta;\varepsilon), \tag{15}$$

where $u = w\,J^{\frac{1}{2}}$ and $\Lambda = \lambda^{-1}$. We define the operators $A(\varepsilon)$ and A_i in $A(\varepsilon)=\sum_{i=0}^{\infty} A_i\,\varepsilon^i$ by identifying (15) with

$$A(\varepsilon)\,u = \Lambda u \tag{16}$$

and base a Hilbert space on the inner product and norm

$$(u, v) = \iint\limits_{R_0} u(\xi,\eta)\,v(\xi,\eta)\,d\xi\,d\eta, \qquad \|u\|^2 = (u, u).$$

The bound on the A_i required by the RELLICH-NAGY theory,

$$\|A_i\| < K_1\,c_1^n, \qquad K_1 \text{ and } c_1 \text{ constants}, \qquad \|A_i\| \equiv \limsup_{\|u\|=1} |(A\,u, u)| \tag{17}$$

can then be deduced from (15) by estimating the integrals involved, such as

$$\Gamma^4 \equiv \iint\limits_{R_0} \iint\limits_{R_0'} G_0^4(\xi,\eta;\xi',\eta')\,d\xi'\,d\eta'\,d\xi\,d\eta. \tag{18}$$

The remainder of the proof consists largely of showing that convergence in the mean for the eigenfunction series implies their uniform convergence and that the formal perturbation calculations from the differential equations yield the solutions whose existence has been demonstrated.

We now mention three generalizations of the principle result:

a) Even if the unperturbed eigenvalue $\lambda^{(0)}$ has multiplicity μ, $1<\mu<\infty$, the appropriate formal calculations can be shown to be valid for sufficiently small $|\varepsilon|$. Explicit estimates require further information on how the several perturbed eigenvalues diverge from $\lambda^{(0)}$ as $|\varepsilon|$ increases. An easily handled case is that considered by NAGY [12] where an unperturbed eigenvalue $\Lambda^{(0)}$ of multiplicity μ splits at order ε into μ simple eigenvalues.

b) Using an appropriate Green's function, one can show without difficulty that our results hold if, in (1), the Dirichlet boundary condition $W=0$ on B_ε is replaced by either the Neumann condition $\frac{\partial W}{\partial n}=0$ on B_ε or, if B_ε is taken to be composed of two arcs $B_\varepsilon^{(1)}$ and $B_\varepsilon^{(2)}$, by the mixed boundary condition $\frac{\partial W}{\partial n}=0$ on $B_\varepsilon^{(1)}$, $W=0$ on $B_\varepsilon^{(2)}$.

c) One obtains a better convergence estimate than (12) by adapting the work in [12] to show that if, in addition to (17), the inequality

$$\|A(\varepsilon) - A_0\| < M(\varepsilon) \tag{19}$$

holds, then the series for Λ and u of (16) converge if

$$|\varepsilon| < c_1^{-1}, \qquad M(\varepsilon) < \frac{d}{4}. \tag{20}$$

4. Some regions for which the method converges

We shall first demonstrate that the MP method converges for boundaries B_ε which are analytic near-circles. For this class of curves we prove in the Appendix that an inequality of the form (11) is satisfied; this shows that the formal calculations are appropriate for $|\varepsilon|$ sufficiently small. For simple eigenvalues, an estimate can be obtained from (20) with the aid of the following result of WARSCHAWSKI [*14*]: Suppose that the boundary B of a region R is a closed Jordan curve represented by the equation $r=r(\vartheta)$ where r is a continuous function of period 2π. Suppose further that, with $F(\vartheta)\equiv\frac{r'(\vartheta)}{r(\vartheta)}$, for some δ

$$1\leqq r(\vartheta)\leqq 1+\delta, \quad |F(\vartheta)|\leqq\delta, \quad |F(\vartheta+\tau)-F(\vartheta)|\leqq|\tau|\,\delta. \tag{21}$$

Then if $z=z(\zeta)$ maps $|\zeta|<1$ conformally onto R such that $z(0)=0$ and $z'(0)>0$, the inequality

$$|z'(\zeta)-1|\leqq 2(1+\delta)^2\,4^\delta\exp(\delta^2)-2-\delta\equiv M_0(\delta) \tag{22}$$

holds for $|\zeta|\leqq 1$.

Using this, we see that in the expression

$$\begin{aligned}&\|[A(\varepsilon)-A_0]\,u\|^2\\ &\quad\equiv\iint\limits_{R_0}\Big\{\iint\limits_{R_0'} G(\xi,\eta;\xi',\eta')\left[\left|\frac{dz}{d\zeta}(\xi,\eta)\right|\left|\frac{dz}{d\zeta}(\xi',\eta')\right|-1\right]u(\xi',\eta')\,d\xi'\,d\eta'\Big\}^2 d\xi\,d\eta,\end{aligned}$$

the quantity in the square brackets is less in absolute value than $M_0^2+2M_0\equiv M_1$. In the Appendix it is shown that the remaining part of the integral is less than 1, which yields

$$\|[A(\varepsilon)-A_0]\,u\|^2\leqq M_1^2\,\|u\|^2, \quad \|A(\varepsilon)-A_0\|\leqq M_1, \tag{23}$$

so that the M of (19) may be identified with M_1.

As an illustration, we apply this work to the principal eigenvalue of the ellipse (6), which we write in the form

$$E: r=(1-\varepsilon\cos 2\vartheta)^{-\frac{1}{2}}. \tag{24}$$

Consider $\varepsilon\geqq 0$. As $(1+\varepsilon)^{-\frac{1}{2}}\leqq r\leqq(1-\varepsilon)^{-\frac{1}{2}}$, to use WARSCHAWSKI's theorem we first subject (24) to the mapping $z_1=(1+\varepsilon)^{\frac{1}{2}}z$ which takes E into an ellipse E_1 satisfying (21) if $\delta=2\varepsilon(1-\varepsilon)^{-1}$. Let $z_1=z_1(\zeta)$ map $|\zeta|<1$ onto the interior of E_1; (21) then holds for $z_1'(\zeta)$. This leads to

$$|z'(\zeta)-1|<(M_0+1)\,(1+\varepsilon)^{-\frac{1}{2}}-1\equiv M_0'. \tag{25}$$

The lowest unperturbed eigenvalue, *i.e.* the lowest eigenvalue for a circular membrane, is 5.78, and the next lowest is 14.68, so that, in (20), $d=0.105$. Using M_0' of (25), $M_0=5.8\,\delta+O(\delta^2)$, $\delta=\frac{2\varepsilon}{(1-\varepsilon)}$ and $M=(M_0')^2+2M_0'$, we see that the requirement $M<\frac{d}{4}$ is satisfied when $\varepsilon<0.0012$. From the Appendix, the other requirement of (20) certainly holds for this range of ε. Therefore the principal eigenvalue of the region bounded by the ellipse (6) is an analytic function of ε

provided $0<\varepsilon<0.0012$. Equivalently, since the eccentricity E is related to ε by $E^2=2\varepsilon(1+\varepsilon)^{-1}$, the lowest eigenvalue of the elliptic membrane is an analytic function of the square of its eccentricity provided that its eccentricity is less than about $\frac{1}{20}$.

We remark that, by use of results in [4], it can be shown that the MP procedure is convergent for any membrane bounded by a curve departing slightly and analytically (in the sense made precise in [4]) from some non-singular curve. We note that in any rigorous discussion it is an advantage of the MP method to be able to draw on the large body of research in function theory.

5. Triangular membranes

Exact solutions to the membrane problem are known if the boundary is a rectangle or an isosceles right triangle ([7], p. 754–756), or a regular hexagon or an equilateral triangle [2]. It is natural to hope that these solutions, too, can be perturbed successfully, although one at first would perhaps be pessimistic because of the mappings' branch points at corners of the polygons. We consider here the particular case of a triangle which is almost an isosceles right triangle, and we prove that the perturbation method yields convergent series for the eigenvalues and eigenfunctions. It would seem that the proof could be carried through for any region bounded by straight lines.

Consider a nearly isosceles right triangle T_ε whose angles at $z=1$ and $z=0$ are $\frac{\pi}{4}+\alpha\varepsilon\pi$ and $\frac{\pi}{4}+\beta\varepsilon\pi$ (α and β are constants). The MP procedure requires mapping T_0 onto T_ε where T_0 is a triangle whose angles at $\zeta=0$ and $\zeta=1$ are both $\frac{\pi}{4}$. By the relation

$$z=C_\varepsilon\int_0^h(1-\xi)^{-\frac{3}{4}+\alpha\varepsilon}\,\xi^{-\frac{3}{4}+\beta\varepsilon}\,d\xi,\qquad C_\varepsilon^{-1}\equiv\int_0^1(1-\xi)^{-\frac{3}{4}+\alpha\varepsilon}\,\xi^{-\frac{3}{4}+\beta\varepsilon}\,d\xi,\tag{26}$$

the interior of T_ε is mapped onto the upper half h-plane, the points $z=0$, $z=1$ remaining fixed. With the same fixed points, T_0 is mapped onto the upper h-plane by

$$\zeta=C_0\int_0^h\xi^{-\frac{3}{4}}(1-\xi)^{-\frac{3}{4}}\,d\xi,\tag{27}$$

which completes the originally required mapping in terms of the intermediate complex variable h. As the eigenvalues and eigenfunctions for T_0 are known, the MP method may be used. Integrations will be performed in the h-plane, so we note that if $h=a+ib$ and $\widetilde{F}(h)=F(\zeta)$, then

$$\iint_{T_0}F\,d\xi\,d\eta=\iint_{b\geq0}\widetilde{F}\left|\frac{d\zeta}{dh}\right|^2 da\,db=\iint_{b\geq0}\widetilde{F}\,|h|^{-\frac{3}{2}}\,|1-h|^{-\frac{3}{2}}\,da\,db.\tag{28}$$

The convergence theorem requires that $\left|\frac{dz}{d\zeta}\right|$ satisfy (11) where $\left|\frac{dz}{d\zeta}\right|=C_\varepsilon C_0^{-1}A^\varepsilon$, $A\equiv|h|^\beta\,|1-h|^\alpha$. To demonstrate (11), we first consider the formal expansion

$$A^\varepsilon=\sum_{n=0}^{\infty}\varepsilon^n\widetilde{H}_n,\qquad \widetilde{H}_n\equiv\frac{(\ln A)^n}{n!}.$$

A bound on $\iint\limits_{b \geqq 0} \widetilde{H}_n^4 \left|\frac{d\zeta}{dh}\right|^2 da\, db$ can be obtained by dividing $b \geqq 0$ into region I where $A \leqq 1$ and region II where $A \geqq 1$, and by observing that for any $\gamma > 0$,

$$(\ln x)^{4n} \leqq \left(\frac{4n}{e\gamma}\right)^{4n} x^{-\gamma}, \qquad 0 < x \leqq 1, \tag{29}$$

$$\leqq \left(\frac{4n}{e\gamma}\right)^{4n} x^{\gamma}, \qquad 1 \leqq x < \infty. \tag{30}$$

Stirling's formula, as in [*15*], p. 251—253, gives $(n!)^4 \geqq 4\pi^2(n+1)^{4n+2} e^{-4(n+1)}$, which shows, using (30), that for some constant K

$$\iint\limits_{\mathrm{II}} \widetilde{H}_n^4 \left|\frac{d\zeta}{dh}\right|^2 da\, db \leqq K\left(\frac{4}{\gamma}\right)^{4n} \iint\limits_{\mathrm{II}} |h|^{-\frac{3}{2}+\gamma\beta} |1-h|^{-\frac{3}{2}+\gamma\alpha} da\, db. \tag{31}$$

Application of (29) shows that an inequality like (31) holds over region I. Choosing γ small enough to insure convergence of the bounding integrals, from (28) we have $\iint\limits_{T_0} H_n^4\, d\xi\, d\eta \leqq K\left(\frac{4}{\gamma}\right)^{4n}$. Similarly, from (26), the coefficient of ε^n in $\frac{C_0}{C_\varepsilon}$ is less than a constant multiple of $\left(\frac{4}{\gamma}\right)^n$. So is the coefficient of ε^n in $\frac{C_\varepsilon}{C_0}$, for $0 < \frac{C_0}{C_\varepsilon} \leqq 1$ implies that the two series have the same radius of convergence. The desired inequality, $\iint\limits_{T_0} \varphi_n^4\, d\xi\, d\eta < K\left(\frac{8}{\gamma}\right)^{4n}$, now follows without difficulty.

We have therefore proved convergence of the method for the triangle T_ε. In particular, if $|\varepsilon|$ is sufficiently small, any eigenfunction may be represented by a power series (5) which converges uniformly for all (ξ, η) in T_0. With the series $\xi(x, y; \varepsilon) = x + \varepsilon\,\xi_1(x, y) + \cdots$ and $\eta(x, y; \varepsilon) = y + \varepsilon\,\eta_1(x, y) + \cdots$, derivable from the mapping and the definition $W(x, y; \varepsilon) = w[\xi(x, y; \varepsilon), \eta(x, y; \varepsilon)]$, one can find expressions for the terms in the expansion

$$W(x, y; \varepsilon) = W_0(x, y) + \varepsilon W_1(x, y) + \cdots \tag{32}$$

of the eigenfunction in the original independent variables; *e.g.*,

$$W_0 = w_0, \quad W_1 = \xi_1\left(\frac{\partial w_0}{\partial x}\right) + \eta_1\left(\frac{\partial w_0}{\partial y}\right) + w_1. \tag{33}$$

The expressions for succeeding terms in (32) contain progressively higher order derivatives of terms W_i of progressively higher subscript, so unless all the terms of (5) are infinitely differentiable, (32) cannot converge uniformly. Even if it does converge uniformly, (31) compares unfavorably with (32): as is easily seen, $\frac{\partial \xi_1}{\partial x}$ (say) is singular at the triangle's corners so — from (33) — w_1 would not even be once differentiable while W_1 would be infinitely differentiable.

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Appendix: Inequalities involving $\left|\frac{dz}{d\zeta}\right|$ for mappings of analytic near-circles

In 1946, GOLUSIN proved that the power series in ε giving the mapping $z=z(\zeta;\varepsilon)$ of $|\zeta|\leqq 1$ onto the interior of the analytic near-circle (7) converges if $|\varepsilon|<\varepsilon^*$, uniformly for $|\zeta|\leqq 1$. A lower bound for ε^* is obtained in GOLUSIN's proof ([4], p. 391 ff.). This bound is worked out in detail for the ellipse (6). There is a slight error; the answer should be $\varepsilon^*\geqq 0.0145$. To extend GOLUSIN's result to series (10) for $\left|\frac{dz}{d\zeta}\right|$, we note that, for a fixed ε, $z(\zeta;\varepsilon)$ represents a conformal mapping and so is an analytic function of ζ if $|\zeta|<1$. Since a function separately analytic in each of several variables is analytic in all together, if δ is any small positive constant, then z can be represented by an expansion

$$z(\zeta;\varepsilon)=\sum_{m,n} a_{mn}\,\varepsilon^m\zeta^n \tag{34}$$

which converges uniformly for $|\zeta|\leqq 1-\delta$, $|\varepsilon|\leqq\varepsilon^*-\delta$. But since $z=z(\zeta;\varepsilon^*-\delta)$ maps a circle onto an analytic curve, $z(\zeta;\varepsilon^*-\delta)$ is an analytic function of ζ for $|\zeta|\leqq 1$, not just for $|\zeta|<1$ ([8], p. 186). Therefore series (34) for $z(\zeta,\varepsilon^*-\delta)$ converges for $|\zeta|\leqq 1$, (34) converges uniformly for $|\zeta|\leqq 1$ and $|\varepsilon|\leqq\varepsilon^*-\delta$, and for these values of ζ and ε, we may obtain, from (34), series for $\frac{dz}{d\zeta}$, $\overline{\frac{dz}{d\zeta}}$, and $J\equiv\left|\frac{dz}{d\zeta}\right|^2$. Assuming no further restriction is needed to insure that $|J-1|<1$, (10) converges for $|\varepsilon|\leqq\varepsilon^*-\delta$, uniformly in $\xi^2+\eta^2\leqq 1$. Consequently lim sup $(\xi^2+\eta^2\leqq 1,\ n\to\infty)$ $|\varphi_n|^{\frac{1}{n}}=(\varepsilon^*-\delta)^{-1}$, so that there exists a constant b such that

$$\iint_{\xi^2+\eta^2<1}\varphi_n^4\,d\xi\,d\eta<b\,(\varepsilon^*)^{-4n},$$

which is the inequality required in the convergence proof.

Bound on a Green's function integral. If G is the Green's function of ∇^2 vanishing on $\xi^2+\eta^2=1$, we give the upper bound for the quantity $I=\iint_{R_0}\iint_{R_0'} G^2(\xi,\eta;\xi',\eta')\,d\xi'\,d\eta'\,d\xi\,d\eta$ required under (22). As in [3], p. 377, if $\varrho^2=\xi^2+\eta^2$ and $(\varrho')^2=(\xi')^2+(\eta')^2$, then $G=-\frac{1}{2}\pi(\ln\varrho_1+\ln\varrho_2\varrho')$. Here ϱ_1 and ϱ_2 are the distances of (ξ,η) from (ξ',η') and from its reflection in the unit circle. It is easily seen from a figure and use of the law of cosines that $1-\varrho\varrho'\leqq\varrho_2\varrho'\leqq 1+\varrho\varrho'\leqq 2$ and $|\varrho-\varrho'|\leqq\varrho_1\leqq\varrho+\varrho'\leqq 2$. Using $(\ln x)^2\leqq 2.3\,x^{-\frac{1}{2}}$ for $0<x\leqq 1$ and Schwarz's inequality, we find $I^2\leqq(4\pi^2)^{-1}(\pi^2)\,(2.3)\,[I_1+I_2+(I_1I_2)^{\frac{1}{2}}]$ where

$$I_1=2\int_0^1\int_{\varrho'}^1\frac{\varrho\,\varrho'}{(\varrho-\varrho')^{\frac{1}{2}}}\,d\varrho\,d\varrho'=0.762 \quad\text{and}\quad I_2=\int_0^1\int_0^1\frac{\varrho\,\varrho'}{(1-\varrho\,\varrho')^{\frac{1}{2}}}\,d\varrho\,d\varrho'=0.374,$$

so that $I^2<0.99$ and $I<1$.

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On Lancaster's Decomposition of a Matrix Differential Operator

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1. In the theory of linear vibrations of damped systems a matrix differential equation of the form

$$\left(A \frac{d^2}{dt^2} + B \frac{d}{dt} + C\right) \xi(t) = \zeta(t), \tag{1}$$

with $(n \times n)$ matrices A, B, C and $(n \times 1)$ vectors $\xi(t), \zeta(t)$, usually plays a central role. Closely connected with this equation are the non-linear eigenvalue problems

$$(A\lambda^2 + B\lambda + C)\,\xi = 0, \qquad \eta(A\lambda^2 + B\lambda + C) = 0, \tag{2}$$

where ξ, η are the eigenvectors and λ is the corresponding eigenvalue.

In this connection P. LANCASTER [*1*] has recently discovered a decomposition of the operator in (1) which holds under fairly general conditions and appears to be very useful.

Since LANCASTER'S proof is imbedded in a rather cumbersome general investigation and is not very accessible, I shall give an independent formulation in Section 2 and a very simple proof of this result in Section 3.

The decomposition in question uses only one half of the right and left eigenvectors of (2). In Sections 4, 5 of this note I give simple expressions for the remaining eigenvectors and use these expressions in Section 6 to present LANCASTER'S solution of (1) by quadratures in an alternate form which may in some cases be easier to handle than the expression given by LANCASTER.

It has been found convenient to use notations different from LANCASTER'S.

2. **Theorem.** *Consider the differential operator*

$$A D^2 + B D + C, \tag{3}$$

where A, B, C are constant $(n \times n)$-matrices and D is the differentiation operator with respect to t. Assume that the $2n$ roots of the equation

$$|A\sigma^2 + B\sigma + C| = 0$$

can be distributed in two sets, of n roots each,

$$S_1 = (\sigma_1', \ldots, \sigma_n'), \qquad S_2 = (\sigma_1'', \ldots, \sigma_n''),$$

such that none of the roots in S_1 equals any root in S_2, and such that there are n independent right eigenvectors $\xi_1, \ldots, \xi_n$ corresponding to the $\sigma_1', \ldots, \sigma_n'$ and n

independent left eigenvectors $\eta_1, \dots, \eta_n$, *corresponding to* $\sigma_1'', \dots, \sigma_n''$. *Put*

$$(4) \qquad \Lambda_1 = \begin{pmatrix} \sigma_1' & 0 & \dots & 0 \\ 0 & \sigma_2' & \dots & 0 \\ . & . & . & . \\ 0 & 0 & \dots & \sigma_n' \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} \sigma_1'' & 0 & \dots & 0 \\ 0 & \sigma_2'' & & \\ . & . & . & . \\ 0 & 0 & \dots & \sigma_n'' \end{pmatrix},$$

and define two $(n \times n)$*-matrices* Q, R *by*

$$(5) \qquad R_1 = (\xi_1, \dots, \xi_n), \quad L_2 = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}.$$

Then

$$(6) \qquad A D^2 + B D + C = L_2^{-1} (I D - \Lambda_2) L_2 A R_1 (I D - \Lambda_1) R_1^{-1}.$$

3. **Proof.** Multiplying (3) from the right by R_1 and from the left by L_2, we reduce the general case to the case where both R_1 and L_2 are unity matrices. Then each ξ_k is the k^{th} coordinate unity vector $e_k = (\delta_{\mu k})$ corresponding to σ_k', while η_k is also $e_k' = (\delta_{k\mu})$ corresponding to σ_k''. We have then

$$(A \sigma_i'^2 + B \sigma_i' + C) e_i = 0 \qquad (i = 1, \dots, n),$$

and, taking here the k^{th} row and putting

$$A = (a_{\mu\nu}), \quad B = (b_{\mu\nu}), \quad C = (c_{\mu\nu}),$$

$$(7) \qquad a_{ki} \sigma_i'^2 + b_{ki} \sigma_i' + c_{ki} = 0 \qquad (i, k = 1, \dots, n).$$

In the same way we have

$$e_k' (A \sigma_k''^2 + B \sigma_k'' + C) = 0,$$

or, considering the i^{th} column,

$$(8) \qquad a_{ki} \sigma_k''^2 + b_{ki} \sigma_k'' + c_{ki} = 0 \qquad (i, k = 1, \dots, n).$$

From (7) and (8) we have

$$(9) \qquad b_{ki} = -a_{ki} (\sigma_i' + \sigma_k''), \quad c_{ki} = a_{ki} \sigma_i' \sigma_k'';$$

this can be written as

$$(10) \qquad B = -(\Lambda_2 A + A \Lambda_1), \quad C = \Lambda_2 A \Lambda_1,$$

and these relations give (6). Our theorem is proved.

4. Under the hypotheses of our theorem it follows from the Fredholm alternative that to the roots σ_ν'' in S_2 correspond n linearly independent right eigenvectors $\xi_1^{(1)}, \dots, \xi_n^{(1)}$, and to the σ_ν' in S_1 n linearly independent left eigenvectors $\eta_1^{(1)}, \dots, \eta_n^{(1)}$. Further, under our hypotheses, A is not singular.

To obtain the $\xi_\nu^{(1)}$ observe that if we write

$$(11) \qquad L_2 A R_1 = T^{-1}, \quad R_1^{-1} A^{-1} L_2^{-1} = T = (t_{\mu\nu}),$$

to the decomposition (6) corresponds the decomposition

$$A\lambda^2 + B\lambda + C = L_2^{-1}(I\lambda - \Lambda_2)\, T^{-1}(I\lambda - \Lambda_1)\, R_1^{-1}. \tag{12}$$

A right eigenvector $\xi_\nu^{(1)}$ corresponding to σ_ν'' is obtained from the equation

$$L_2^{-1}(I\sigma_\nu'' - \Lambda_2)\, T^{-1}(I\sigma_\nu'' - \Lambda_1)\, R_1^{-1}\xi_\nu^{(1)} = 0,$$

or, putting

$$T^{-1}(I\sigma_\nu'' - \Lambda_1)\, R_1^{-1}\xi_\nu^{(1)} = \hat{\xi}, \tag{13}$$

$$(I\sigma_\nu'' - \Lambda_2)\,\hat{\xi} = 0. \tag{14}$$

Then (14) is satisfied by $\hat{\xi} = e_\nu = (\delta_{\mu\nu})$, and we obtain from (13)

$$\xi_\nu^{(1)} = R_1(I\sigma_\nu'' - \Lambda_1)^{-1}\, T e_\nu. \tag{15}$$

5. From now on we use the notation $A^{(\nu)}$ for the column vector formed by the ν^{th} column of A. Observe that we have then from the law of multiplication of matrices

$$(AB)^{(\nu)} = A\cdot B^{(\nu)}. \tag{16}$$

Further, we introduce the $(n\times n)$-matrix

$$\Omega = \left(\frac{t_{\mu\nu}}{\sigma_\nu'' - \sigma_\mu'}\right) \qquad (\mu, \nu = 1, \ldots, n). \tag{17}$$

We have then from (15)

$$T e_\nu = T^{(\nu)}, \qquad (I\sigma_\nu'' - \Lambda_1)^{-1}\, T^{(\nu)} = \Omega^{(\nu)},$$

$$\xi_\nu^{(1)} = (R_1\Omega)^{(\nu)}. \tag{18}$$

Here $R_1\Omega$ is not singular, and therefore (18) gives for any multiple root σ_ν'' independent eigenvectors equal in number to the multiplicity of σ_ν''. We see that the matrix

$$R_2 = R_1\Omega \tag{19}$$

consists of n linearly independent right eigenvectors corresponding to the eigenvalues of S_2 and ordered in the same way as in S_2 and Λ_2. A completely symmetric discussion gives us the matrix

$$L_1 = -\Omega L_2. \tag{20}$$

6. The differential equation (1) becomes, in virtue of (6) and (11),

$$L_2^{-1}\left(I\frac{d}{dt} - \Lambda_2\right) T^{-1}\left(I\frac{d}{dt} - \Lambda_1\right) R_1^{-1}\xi(t) = \zeta(t), \tag{21}$$

or, putting

$$T^{-1}\left(I\frac{d}{dt} - \Lambda_1\right) R_1^{-1}\xi(t) = \alpha(t), \tag{22}$$

$$L_2^{-1}\left(I\frac{d}{dt} - \Lambda_2\right)\alpha(t) = \zeta(t). \tag{23}$$

An integral of (23) is given, as is immediately verified by differentiation, by

$$\alpha(t) = e^{\Lambda_2 t} \int_c^t e^{-\Lambda_2 \tau} L_2 \zeta(\tau)\, d\tau, \tag{24}$$

and we have in the same way from (22) and (24)

$$\xi(t) = R_1 e^{\Lambda_1 t} \int_c^t e^{-\Lambda_1 \tau'} T \alpha(\tau')\, d\tau',$$

$$\xi(t) = R_1 e^{\Lambda_1 t} \int_c^t \int_c^{\tau'} e^{-\Lambda_1 \tau'} T e^{\Lambda_2 \tau'} e^{-\Lambda_2 \tau} L_2 \zeta(\tau)\, d\tau\, d\tau'. \tag{25}$$

Interchanging the order of integrations by Dirichlet's rule, we have

$$\xi(t) = R_1 e^{\Lambda_1 t} \int_c^t K(\tau) e^{-\Lambda_2 \tau} L_2 \zeta(\tau)\, d\tau, \tag{26}$$

$$K(\tau) = \int_\tau^t e^{-\Lambda_1 \tau'} T e^{\Lambda_2 \tau'}\, d\tau'. \tag{27}$$

If we consider in the matrix (27) the element in the μ^{th} row and the ν^{th} column, we obtain for the indefinite integral, denoting by $t_{\mu\nu}$ the corresponding component of T,

$$\frac{1}{\sigma''_\nu - \sigma'_\mu} e^{-\sigma'_\mu t} t_{\mu\nu} e^{\sigma''_\nu t},$$

and this is, by (17), the corresponding element of the matrix

$$e^{-\Lambda_1 t} \Omega e^{\Lambda_2 t}.$$

We have therefore from (27) and (26)

$$K(\tau) = e^{-\Lambda_1 t} \Omega e^{\Lambda_2 t} - e^{-\Lambda_1 \tau} \Omega e^{\Lambda_2 \tau},$$

$$\xi(t) = R_1 \int_c^t \left(\Omega e^{\Lambda_2 (t-\tau)} - e^{\Lambda_1 (t-\tau)} \Omega\right) L_2 \zeta(\tau)\, d\tau, \tag{28}$$

or, using (19) and (20),

$$\xi(t) = \int_c^t \left(R_2 e^{\Lambda_2 (t-\tau)} L_2 + R_1 e^{\Lambda_1 (t-\tau)} L_1\right) \zeta(\tau)\, d\tau. \tag{29}$$

(28) is equivalent to the formula (39) in LANCASTER's paper [*1*].

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The Effect of a Very Strong Magnetic Cross-Field on Steady Motion through a Slightly Conducting Fluid: Three-Dimensional Case

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1. Introduction

We continue our discussion (LUDFORD 1961) of the steady flow of a slightly conducting, incompressible, inviscid fluid past a fixed obstacle, when there is a very strong magnetic field $\boldsymbol{H}_0$ applied perpendicular to the free stream $\boldsymbol{U}_0$. Previously, we considered plane flow past a cylinder of small conductivity; here we treat a finite insulator of general shape, whose permeability μ will again be taken equal to that of the fluid. The present paper is, however, self-contained.

Again the relative change in the magnetic field, due to motion-induced current, is small, on account of the smallness of the conductivity σ, and may be neglected. On the other hand, the fluid motion is no longer close to the potential flow which obtains for $\sigma = 0$, since the current, though small, experiences large ponderomotive force due to the strength of $\boldsymbol{H}_0$. In the limit $\sigma \to 0$, $\sigma H_0^2 \to \infty$ the velocity is only disturbed in the direction of the applied magnetic field and all quantities are constant in this direction.

The limit equations possess a multiplicity of solutions, none of which satisfies the boundary conditions completely. The simplest is in general not the correct one. Some care is needed in deriving the correct limit equations, moreover. It is easily shown that if terms containing the charge density ϱ_e are neglected (magnetohydrodynamic approximation) the resulting limit equations lead to even greater indeterminateness: they ensure that the local electric field is everywhere zero and thereby render ineffective the boundary condition of no current flow across the surface of the body. Consequently, the fluid velocity across the magnetic lines of force, and hence the vorticity along them, is undetermined. In fact ϱ_e is zero in the present approximation, as it is automatically in the plane case. But it is essential to know that it is accurately zero and not just negligible. The situation is similar to that in electrostatics, where the electric field in a dielectric is not determined until the volume charge is specified. In the present case it is sufficient to know that the fluid is uncharged upstream since this condition persists throughout the motion of a fluid element.

It is at this point that our problem differs from STEWARTSON's (1956), who has considered slow flow with σ and H_0 arbitrary. Provided the ϱ_e-terms are neglected the problems are identical*: the same limiting values of the parameters

* Except for the type of body considered.

R_M and N [see equations (2a, b)] are involved. We then agree with his conclusion that the flow is not determined by the steady state equations. In his case it is inappropriate to include the ϱ_e-terms, since they are of higher order: the new parameter β^2 [see equation (2b)] thereby introduced is smaller still than R_M. In fact $\beta^2/R_M = \frac{(1/c^2\mu\sigma)}{(a/U_0)}$ is the ratio of the charge relaxation time to the transit time of the fluid past the body. In our case it is large; in his it is small. To determine his flow he considers how the motion is set up from rest.

This is not necessarily the reason why STEWARTSON's steady flow is so radically different from ours. Another possible explanation is that the bodies are different: his is a perfect conductor, ours a perfect insulator. It would be instructive to have the solution of STEWARTSON's problem for a dielectric sphere and of ours for a perfect conductor.

At this stage we know the correct form of the limit equations, but still have to select the correct solution from among the multiplicity of possible ones. We are in fact at what was the starting point in the plane case, and can immediately show that certain quantities, such as the disturbance velocity perpendicular to the magnetic lines of force (in the y-direction), and changes along them are zero in the limit. To determine the solution we must, as before, compress the flow in the y-direction so as to magnifiy vanishingly small quantities; since all of the latter can now be identified, this leads to a definite transformation of variables and a set of "boundary layer" equations.

Fortunately these equations are once more linear. They take into account the inertia forces in the fluid, which play a critical role at large y-distances in dispersing the disturbance due to the body and thereby controlling the flow around it. In this manner the correct solution of the previous limit equations is selected and the way in which the pressure becomes infinite at each finite point determined.

The pressure and hence the forces on the body are of order $\sqrt{\sigma}\, H_0$ (*i.e.* $\sqrt{N}$). The drag has been computed for a general ellipsoid with one axis along $\boldsymbol{U}_0$ and another along $\boldsymbol{H}_0$. As the third axis becomes indefinitely long the drag coefficient tends to a value considerably less than that of the elliptic cylinder treated previously (LUDFORD 1961). Unfortunately, no explicit results have been obtained for a body which is not symmetric with respect to the x, z-plane. The underlying integral equation has singularities which appear to be intractable.

2. The Equations of Steady Motion and Their Limiting Form

If the fluid velocity is denoted by $U_0\boldsymbol{v}$, the pressure by $\varrho_0 U_0^2 p$, the magnetic field by $H_0\boldsymbol{H}$, the electric field by $\mu U_0 H_0 \boldsymbol{E}$, and the coordinates by $(a x,\ a y,\ a z)$ — where a is a length — then the flow is governed by the equations

$$\boldsymbol{v}\cdot\operatorname{grad}\boldsymbol{v} = -\operatorname{grad} p + \frac{1}{A^2}(\varrho_e\boldsymbol{E} + \boldsymbol{J}\times\boldsymbol{H}), \tag{1a}$$

$$\boldsymbol{J} - \varrho_e\boldsymbol{v} = R_M(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{H}), \tag{1b}$$

$$\operatorname{div} v = 0, \qquad \operatorname{curl}\boldsymbol{E} = 0, \qquad \operatorname{div}\boldsymbol{H} = 0,$$

where

$$\varrho_e = \beta^2 \operatorname{div}\boldsymbol{E}, \qquad \boldsymbol{J} = \operatorname{curl}\boldsymbol{H}.$$

All quantities are dimensionless, the charge density being $(H_0/a\,U_0)\,\varrho_e$, and the parameters are

$$\begin{aligned} A &= \frac{U_0}{H_0}\sqrt{\frac{\varrho_0}{\mu}} && \text{(Alfvén number)},\\ R_M &= U_0\,a\,\mu\,\sigma && \text{(magnetic Reynolds number)},\\ \beta &= \frac{U_0}{c}, && \text{where } c^2 = 1/\mu\,\varepsilon. \end{aligned} \tag{2a}$$

Fig. 1 shows the choice of axes.

In equations (1) the terms containing ϱ_e are usually small compared to others, and in the so-called magnetohydrodynamic approximation they are omitted (GOLDSTEIN 1960, p. 60). However, the argument fails in the case of vanishingly small conductivity, and their omission in the present problem leads to indeterminateness in the electric field through the charge density ϱ_e. The latter is in fact zero to the accuracy used here. In the plane problem considered previously (LUDFORD 1961) it was zero automatically.

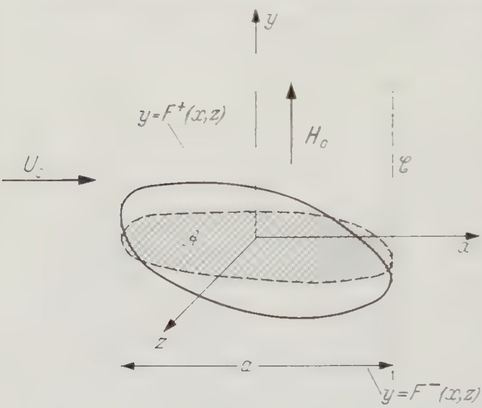

Fig. 1. Notation and axes

Take the divergence of the conduction equation (1b) and set $R_M = 0$. Then $\boldsymbol{v}\cdot\operatorname{grad}\varrho_e = 0$, so that ϱ_e is constant on streamlines. But the incident flow is uncharged. Hence $\varrho_e = 0$, correct to $O(1)$ in R_M, throughout the flow. In other words, as far as the electromagnetic field is concerned the fluid is non-conducting, and for any such material the elemental charge is conserved and must be specified in advance. Here it is zero since each fluid element originates at infinity upstream, where it is uncharged.

It follows that $\boldsymbol{E}$ is a potential field which may be written in the form $-\boldsymbol{k} + \operatorname{grad}\varphi$ where

$$\nabla^2\varphi = 0; \tag{3}$$

then φ is a disturbance potential. Moreover, $\boldsymbol{H}$ is a potential field in the limit; and when the permeability of the body is the same as that of the fluid this leads to $\boldsymbol{H} = \boldsymbol{j}$: the applied magnetic field is undisturbed.

Now consider the momentum equation (1a) which, on substituting for $\boldsymbol{J}$ from equation (1b), becomes

$$\boldsymbol{v}\cdot\operatorname{grad}\boldsymbol{v} = -\operatorname{grad}p + N[\varkappa(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{H}) + (\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{H})\times\boldsymbol{H}],$$

where

$$N = \frac{R_M}{A^2} = \frac{a\,\mu^2 H_0^2\,\sigma}{\varrho_0\,U_0} \tag{2b}$$

and $\varkappa = \varrho_e/R_M$. Using the results just given for $\boldsymbol{E}$ and $\boldsymbol{H}$, we obtain for the component equations

$$\begin{aligned} \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{\partial p}{\partial x} + N\left[\varkappa\left(\frac{\partial\varphi}{\partial x} - w\right) - \left(\frac{\partial\varphi}{\partial z} + u\right)\right],\\ \frac{\partial v}{\partial x} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{\partial p}{\partial y} + N\varkappa\frac{\partial\varphi}{\partial y},\\ \frac{\partial w}{\partial x} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z} + N\left[\varkappa\left(\frac{\partial\varphi}{\partial z} + u\right) - \left(\frac{\partial\varphi}{\partial x} - w\right)\right], \end{aligned} \tag{4}$$

where u, v, w are the components of the disturbance velocity $\boldsymbol{v}-\boldsymbol{i}$. When we speak of a very strong applied magnetic field we mean that N is large, even though σ is small. The equations then reduce to

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{p}{N}\right) &= \varkappa\left(\frac{\partial \varphi}{\partial x} - w\right) - \left(\frac{\partial \varphi}{\partial z} + u\right),\\
\frac{\partial}{\partial y}\left(\frac{p}{N}\right) &= \varkappa \frac{\partial \varphi}{\partial y},\\
\frac{\partial}{\partial z}\left(\frac{p}{N}\right) &= \varkappa\left(\frac{\partial \varphi}{\partial z} + u\right) + \left(\frac{\partial \varphi}{\partial x} - w\right),
\end{aligned} \tag{5}$$

if we allow for the possibility that p becomes large with N. To these must be added the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{6}$$

and the divergence of (1b),

$$(1+u)\frac{\partial \varkappa}{\partial x} + v\frac{\partial \varkappa}{\partial y} + w\frac{\partial \varkappa}{\partial z} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \tag{7}$$

where the right-hand side has been given its limiting value.

3. Solution of the Limit Equations

The following argument is similar to that used in the plane case. We assume that a line parallel to the y-axis cuts the surface of the body in at most two points.

First neglect quadratic terms. Then equations (5) give

$$p = N f(x, z), \qquad u = -f_x - \varphi_z, \qquad w = -f_z + \varphi_x,$$

so that (6) yields

$$v = y(f_{xx} + f_{zz}) + g(x, z);$$

here f and g are arbitrary functions of x and z alone, each of which may be different in the regions vertically above and below the body.

It is clearly not possible to ensure that conditions are undisturbed at infinity in the y-direction. However, it is reasonable to assume that they are only disturbed in a finite manner, *i.e.*,

$$f_{xx} + f_{zz} = 0 \qquad \text{for all } x, z.$$

According to Liouville's theorem f is constant, and if the pressure is not disturbed at infinity in a plane $y=$const. then f is identically zero.

A similar conclusion can be drawn for φ from equation (3) once it is known that φ is independent of y. The latter follows from the boundary condition that no current flows across the surface of the body. For, according to (1b), the normal component of the local electric field in the fluid,

$$\left(\frac{\partial \varphi}{\partial x} - w, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} + u\right) = \left(0, \frac{\partial \varphi}{\partial y}, 0\right),$$

must then be zero at the body; so that $\partial\varphi/\partial y$ is a harmonic function which vanishes on the body. It is therefore identically zero.

Finally, since u and w are now everywhere zero, equation (7) shows that $\varkappa$ is independent of x and hence zero. This ensures that we have in fact the exact solution of equations (5), (6), and (7).

The complete solution is

$$\frac{p}{N} = u = w = \varphi = \varkappa = 0, \qquad v = g(x, z), \tag{8}$$

where for a point within the circumscribing cylinder $\mathscr{C}$ the boundary condition on velocity at the surface $y = F^{\pm}(x, z)$ of the body (see Fig. 1) gives

$$g(x, z) = \begin{cases} F_x^+(x, z), & y \geqq F^+(x, z) \quad \text{in } \mathscr{C}, \\ F_x^-(x, z), & y \leqq F^-(x, z) \quad \text{in } \mathscr{C}. \end{cases}$$

This violates the conditions at infinity: v does not tend to zero as $y \to \pm\infty$. Outside $\mathscr{C}$ these conditions can be satisfied, by setting

$$g(x, z) = 0.$$

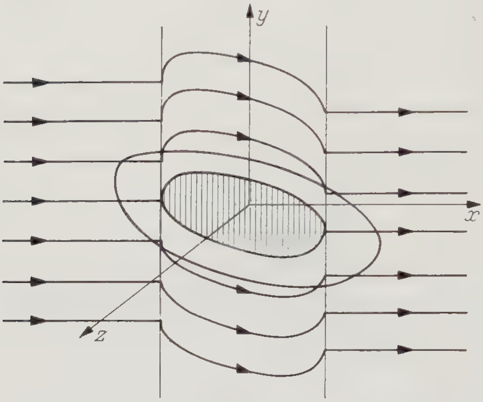

Fig. 2. Flow in plane z = const. for plausible (but incorrect) solution of limit equations

However, it will turn out that this is not in general the correct choice (see end of Section 6).

Only the vertical component of velocity is disturbed: vertical columns of fluid slip freely in the direction of the applied magnetic field on being displaced by the obstacle during their uniform forward motion (see Fig. 2).

The electrical field inside the body is a potential field, and continuity of its tangential component across the surface shows that it is uniform. Hence $\boldsymbol{E} = -\boldsymbol{k}$ throughout space and any difference in dielectric constant between the fluid and the body results in a surface charge of total amount zero. [This is also the solution when the fluid is replaced by a vacuum.]

On the assumption that the pressure and electric field are continuous within the fluid, we have been led to the solution (8) when the body is non-conducting. To be sure, v is not continuous in general, but this is allowable within the framework of the theory of discontinuous solutions (VON MISES 1958).

The situation is quite different when the body is, for example, a perfect conductor: the electric field is the same as when the fluid is replaced by a vacuum, so that φ is no longer zero (for a sphere $\varphi = z/r^3$). Consequently, there is now motion-induced space charge $\varkappa$, see equation (7), arising from the component of vorticity in the direction of the applied magnetic field. [Of course, all this applies only to the linearized equations.]

4. Compressing the y-coordinate

As in the plane case, changes in the y-direction at any fixed point have vanished in the limit $N \to \infty$ and conditions at infinity are violated by v above and below the body. To retain these changes and relieve this violation, the y-coordinate must be compressed as N increases and quantities which vanished previously must be magnified.

Set

$$y=\sqrt{N}Y, \quad p=\sqrt{N}P, \quad u=\frac{U}{\sqrt{N}}, \quad w=\frac{W}{\sqrt{N}}, \quad \varphi=\frac{\Phi}{\sqrt{N}}, \quad \varkappa=\frac{\mathsf{K}}{\sqrt{N}} \tag{9}$$

in equations (4), (6), and (7); assume all new variables and derivatives are of order unity; and let N tend to infinity. Then

$$\begin{aligned} 0&=-\frac{\partial P}{\partial x}-\frac{\partial \Phi}{\partial z}-U, &\quad \frac{\partial v}{\partial x}&=-\frac{\partial P}{\partial Y}, \\ 0&=-\frac{\partial P}{\partial z}+\frac{\partial \Phi}{\partial x}-W, &\quad \frac{\partial U}{\partial x}+\frac{\partial v}{\partial Y}+\frac{\partial W}{\partial z}&=0, \end{aligned} \tag{10}$$

and

$$\frac{\partial \mathsf{K}}{\partial x}=\frac{\partial U}{\partial z}-\frac{\partial W}{\partial x}. \tag{11}$$

To these must be added the new form of equation (3),

$$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial z^2}=0.$$

The occurrence of the term $\partial v/\partial x$ in equations (10) means that the inertia of the fluid is now taken into account. The choice (9) ensures that, in the new variables, the flow is the result of a balance between the inertia forces and the stresses (pressure and Maxwell).

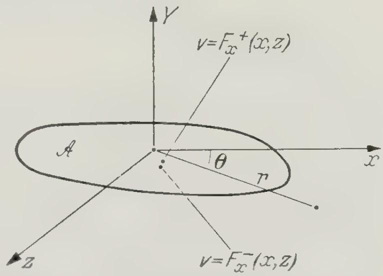

Fig. 3. Boundary conditions on solution of equation (13) in the x, Y, z-space

From this last equation we conclude that $\Phi=0$ in every plane $Y=$const. Then from equations (10) we find

$$U=-\frac{\partial P}{\partial x}, \quad W=-\frac{\partial P}{\partial z}, \tag{12a}$$

and

$$\frac{\partial v}{\partial x}=-\frac{\partial P}{\partial Y}, \quad \frac{\partial v}{\partial Y}=\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial z^2}, \tag{12b}$$

while equation (11) gives $\partial\mathsf{K}/\partial x=0$ so that once more the charge density K is zero.

Eliminate P from equations (12b) to obtain

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial Y^2}=0. \tag{13}$$

The boundary conditions are

$$v=\begin{Bmatrix} F_x^+(x,z) & \text{for} & Y\to+0 \\ F_x^-(x,z) & \text{for} & Y\to-0 \end{Bmatrix} \quad \text{on } \mathscr{A}, \tag{14}$$

where $\mathscr{A}$ is the area bounded by the trace of $\mathscr{C}$ in the x, z-plane (Fig. 1) and represents the whole body in the x, Y, z-space (Fig. 3). Once v has been found the pressure P is uniquely determined by (12b) and then the x- and z-components of velocity from (12a).

The equation (13) will be solved separately above and below the x, z-plane. Each partial solution must satisfy the appropriate one of the conditions (14) on $\mathscr{A}$ and lead to the same values of v and P on the remainder of the x, z-plane.

We may restrict our attention to the half space $Y > 0$ and suppose for the moment that v is prescribed everywhere on the x, z-plane. The complete solution can be built up from the source solution, *i.e.*, the solution having $\delta(x)\,\delta(z)$ for its values in the x, z-plane. Here $\delta(x)$ is the Dirac delta function. In the plane case the source solution is the x-derivative of the solution having the unit step function for boundary values, and the latter is a function of x^3/Y^2 alone which is easily determined by Laplace transform methods. In the present case there is no such solution and we must proceed directly.

5. The Source Solution and its Properties

Since the Fourier transform of $\delta(x)\,\delta(z)$ with respect to x and z is 1 (LIGHTHILL 1958), the function

$$v_0(x, Y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[n x + p z - \sqrt{-n(n^2+p^2)}\,Y\right] d\alpha\, d\beta \qquad (n = 2\pi i \alpha,\ p = 2\pi i \beta),$$

is the source solution of equation (13). For the coefficient of Y we have taken the root with negative real part. The root with positive real part leads to exponentially large values as $Y \to +\infty$.

Of main interest is the corresponding pressure field

$$P_0(x, Y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{n}{\sqrt{-n(n^2+p^2)}} \exp\left[n x + p z - \sqrt{-n(n^2+p^2)}\,Y\right] d\alpha\, d\beta.$$

According to ERDÉLYI *et al.* (1954, p. 17) the β-integral can be evaluated explicitly and

$$P_0(x, Y, z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \sqrt{n}\, e^{n x} K_0\left[2\pi|\alpha|\,(z^2 + n Y^2)^{\frac{1}{2}}\right] d\alpha.$$

The remaining integral can be evaluated for $Y = 0$, which is the case of greatest interest. On writing the resulting integral as the sum of Fourier sine and a Fourier cosine integral, we find from ERDÉLYI *et al.* (1954, pp. 49, 106)

$$\begin{aligned} P_0(x, 0, z) = \Pi(x, z) = \frac{1}{16|z|^{\frac{3}{2}}}\Bigg[&\frac{1}{\Gamma^2(\frac{5}{4})} F\left(\frac{3}{4}, \frac{3}{4}, \frac{1}{2}; -\frac{x^2}{z^2}\right) - \\ &- \frac{2}{\Gamma^2(\frac{3}{4})}\frac{x}{|z|} F\left(\frac{5}{4}, \frac{5}{4}, \frac{3}{2}; -\frac{x^2}{z^2}\right)\Bigg]. \end{aligned} \qquad (15)$$

This is expressible in terms of the Legendre function of order one half (ERDÉLYI *et al.* 1953, p. 128); alternatively complete elliptic integrals may be used*.

$$\begin{aligned} \Pi(x, z) &= \frac{1}{4\sqrt{\pi}\, r^{\frac{3}{2}}} P_{\frac{1}{2}}(-\cos\vartheta) \\ &= \frac{1}{2(\pi r)^{\frac{3}{2}}}\left[2E\left(\left|\cos\frac{\vartheta}{2}\right|\right) - K\left(\left|\cos\frac{\vartheta}{2}\right|\right)\right] = \frac{1}{r^{\frac{3}{2}}}\mathscr{P}(\vartheta) \quad \text{(say)}, \end{aligned} \qquad (16)$$

where r, ϑ are polar coordinates in the x, z-plane (Fig. 3).

* This is easily verified. The only explicit reference the author has found is the index of tables by FLETCHER, MILLER, & ROSENHEAD (1946, p. 238).

Fig. 4 shows a graph of $\mathscr{P}(\vartheta)$ for $0 \leqq \vartheta \leqq \pi$; the function is even in ϑ. For $\vartheta = \pi/2$, *i.e.* on the z-axis,

$$\mathscr{P} = \frac{1}{16\,\Gamma^2(\frac{5}{4})} = 0.0761\,.$$

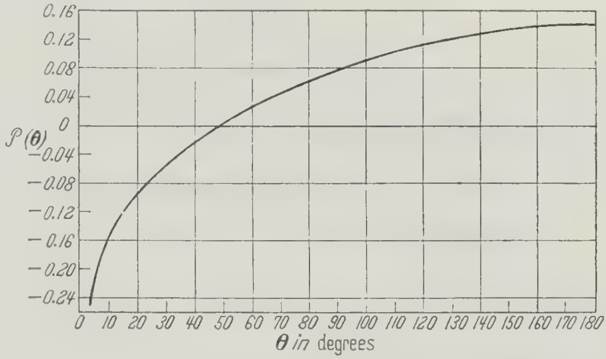

Fig. 4. The pressure function $\mathscr{P}(\vartheta)$ associated with a unit source [equation (16)]

As $\vartheta \to 0$ (positive x-axis)

$$\mathscr{P} \sim \frac{1}{2\pi^{\frac{3}{2}}} \log \sin\vartheta\,,$$

while for $\vartheta = \pi$ (negative x-axis)

$$\mathscr{P} = \frac{1}{4\sqrt{\pi}} = 0.1410\,.$$

The logarithmic infinity* in $\mathscr{P}$ for $\vartheta = 0$ is curious, since in the two-dimensional case of a line source along $x = 0$ there is no downstream influence, *i.e.*, the pressure is zero for $-\pi/2 < \vartheta < \pi/2$ (LUDFORD 1961). However, the present solution checks with this previous result: when integrated over a line of sources, Π gives the correct two-dimensional pressure (Appendix 1).

6. Determination of the Complete Solution

The solution of (13) taking on the values $V(x, z)$ as $Y \to +0$ is

$$v(x, Y, z) = \iint v_0(x - \xi, Y, z - \eta)\, V(\xi, \eta)\, d\xi\, d\eta\,,$$

where the integration is to be taken over all values of ξ and η. There is little to be gained by writing down formulas for the corresponding P, U, and W. All we need is the value of P as $Y \to +0$, and this follows immediately from (16). To satisfy the first of the conditions (14) we set

$$V(x, z) = F_x^+(x, z) \qquad \text{in } \mathscr{A}\,;$$

there remains the determination of V on the remainder of the x, z-plane.

As $Y \to +0$ the pressure is given by

$$P(x, +0, z) = \iint_{\mathscr{A}} \Pi(x - \xi, z - \eta)\, F_x^+(\xi, \eta)\, d\xi\, d\eta + \iint_{C(\mathscr{A})} \Pi(x - \xi, z - \eta)\, V(\xi, \eta)\, d\xi\, d\eta\,, \quad (17)$$

where $C(\mathscr{A})$ is the remainder of the x, z-plane. For the corresponding solution in the half space $Y < 0$, the pressure as $Y \to -0$ is given by the same formula with Π replaced by $-\Pi$ and F_x^+ by F_x^-. Use of the same function $V(x, z)$ in $C(\mathscr{A})$ for the two cases ensures that v takes on the same values as $Y \to \pm 0$ there. To ensure that the pressure [and hence the remaining velocity components] are continuous across $Y = 0$ outside $\mathscr{A}$, we must take

$$\begin{aligned} &\iint_{C(\mathscr{A})} \Pi(x - \xi, z - \eta)\, V(\xi, \eta)\, d\xi\, d\eta \\ &\qquad = -\tfrac{1}{2} \iint_{\mathscr{A}} \Pi(x - \xi, z - \eta)\, [F_x^+(\xi, \eta) + F_x^-(\xi, \eta)]\, d\xi\, d\eta \end{aligned} \qquad (18)$$

* This arises from $K(k)$, which is logarithmically infinite for $k = 1$.

for all values of (x, z) in $C(\mathscr{A})$. Having determined $V(\xi, \eta)$, we may calculate the pressure on the upper and lower surfaces of the body from (17) and its analog.

For a body symmetric about the x, z-plane $F_x^+ = -F_x^-$ and the solution of (18) is

$$V(x, z) = 0 \quad \text{in } C(\mathscr{A}),$$

which is otherwise obvious. The integral equation has so far not proved tractable for any region $\mathscr{A}$ when the body is not symmetric. In any event it is clear that V will be non-zero.

7. The Drag on an Ellipsoid

The drag experienced by the body is

$$D = \varrho_0 U^2 a^2 \sqrt{N} \iint_{\mathscr{A}} [F_x^+(x, z)\, P(x, +0, z) + F_x^-(x, z)\, P(x, -0, z)]\, d x\, d z,$$

and for a body which is symmetric about the x, z-plane this becomes

$$D = 2\varrho_0 U^2 a^2 \sqrt{N} \iint_{\mathscr{A}} d x\, d z \iint_{\mathscr{A}} d\xi\, d\eta\, F_x^+(x, z)\, \Pi(x-\xi, z-\eta)\, F_x^+(\xi, \eta).$$

For an ellipsoid $F^+(x, z) = k\sqrt{1 - x^2 - \alpha^2 z^2}$, where a, ka, a/α are the semi-axes, and $D = \pi \varrho_0 U^2 a^2 \sqrt{N} k^2 I/2\alpha$, where

$$I(\alpha) = \frac{4}{\pi}\alpha \iint_{x^2+\alpha^2 z^2 \leq 1} d x\, d z \iint_{\xi^2+\alpha^2\eta^2 \leq 1} d\xi\, d\eta \frac{\xi\, x\, \Pi(x-\xi, z-\eta)}{\sqrt{(1-x^2-\alpha^2 z^2)(1-\xi^2-\alpha^2\eta^2)}}. \tag{19}$$

$\sqrt{N} I$ is the drag coefficient of the spheroid $k = 1$ based on the area $\pi a \cdot (a/\alpha)$ of $\mathscr{A}$.

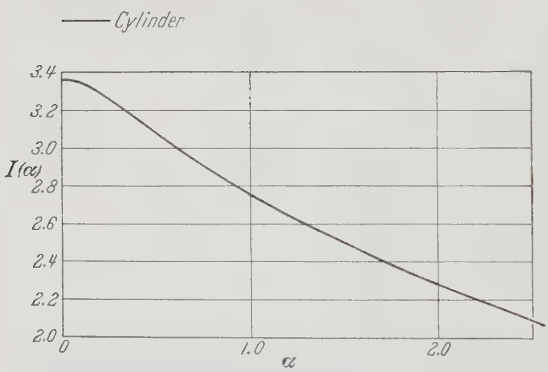

Fig. 5. Values of the integral (19) determining drag on an ellipsoid

The integral has been evaluated numerically (using an eight digit, floating decimal scheme) for $\alpha = 0(0.2)\,1.0(0.5)\,2.5$ and a table is given below. The convergence with decreasing mesh size was slow for small values of α and the results had to be extrapolated*. A graph of I versus α is shown in Fig. 5. The value $I(0) = 3.36$, corresponding to an infinitely long transverse axis, is considerably less than that of a circular cylinder, $B^2(\frac{1}{4}, \frac{1}{2})/\sqrt{18\pi} = 3.66$ (LUDFORD 1961). These values compare with $I(1) = 2.76$ for a sphere:

$\alpha =$	0.0	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5
$I(\alpha) =$	3.37	3.30	3.16	3.02	2.89	2.76	2.48	2.28	2.10

* See Appendix 2 for the reduction of $I(\alpha)$ to a form suitable for numerical computation.

Appendix 1

The pressure exerted at the origin by a line of sources of unit density along $x = x_0 > 0$ is, according to equation (16), given by

$$\overline{P}(x_0) = \int_{-\infty}^{\infty} \Pi(-x_0, z)\, dz = \frac{2}{\sqrt{x_0}} \int_{\pi/2}^{\pi} \mathscr{P}(\vartheta)\, |\sec\vartheta|^{\frac{1}{2}}\, d\vartheta,$$

where $z = -x_0 \tan\vartheta$.

Now $\mathscr{P}(\vartheta)$ may be written as the sum of even and odd functions of $\cos\vartheta$,

$$\mathscr{P}(\vartheta) = \frac{1}{16}\left[\frac{1}{\Gamma^2(\frac{5}{4})} F\left(\frac{3}{4}, -\frac{1}{4}, \frac{1}{2}; \cos^2\vartheta\right) - \frac{2\cos\vartheta}{\Gamma^2(\frac{3}{4})} F\left(\frac{5}{4}, \frac{1}{4}, \frac{3}{2}; \cos^2\vartheta\right)\right].\star$$

Also, with $\cos\vartheta = -\alpha^{\frac{1}{2}}$,

$$\begin{aligned}
&\int_{\pi/2}^{\pi} F\left(\frac{3}{4}, -\frac{1}{4}, \frac{1}{2}; \cos^2\vartheta\right) |\sec\vartheta|^{\frac{1}{2}}\, d\vartheta \\
&\qquad = \frac{1}{2}\int_0^1 F\left(\frac{3}{4}, -\frac{1}{4}, \frac{1}{2}; \alpha\right) \alpha^{-\frac{3}{4}} (1-\alpha)^{-\frac{1}{2}}\, d\alpha = \frac{4}{\sqrt{\pi}}\Gamma^2\left(\frac{5}{4}\right), \\
&\int_{\pi/2}^{\pi} F\left(\frac{5}{4}, \frac{1}{4}, \frac{3}{2}; \cos^2\vartheta\right) |\cos\vartheta|^{\frac{1}{2}}\, d\vartheta \\
&\qquad = \frac{1}{2}\int_0^1 F\left(\frac{5}{4}, \frac{1}{4}, \frac{3}{2}; \alpha\right) \alpha^{-\frac{1}{4}} (1-\alpha)^{-\frac{1}{2}}\, d\alpha = \frac{2}{\sqrt{\pi}}\Gamma^2\left(\frac{3}{4}\right).
\end{aligned} \tag{20}$$

These last results may be obtained by use of the integral representation (Erdélyi *et al.*, 1953, p. 59)

$$F(a, b, c; \alpha) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-t\alpha)^{-a}\, dt,$$

$\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, and interchange of orders of integration. In the former case the hypergeometric function must first be written as

$$1 - \tfrac{3}{8}\alpha \int_0^1 F(\tfrac{7}{4}, \tfrac{3}{4}, \tfrac{3}{2}; \alpha s)\, ds,$$

i.e., as the integral of its derivative.

It follows that

$$\overline{P}(x_0) = \frac{2}{\sqrt{x_0}}\left[\frac{1}{4\sqrt{\pi}} + \frac{1}{4\sqrt{\pi}}\right] = \frac{1}{\sqrt{\pi\, x_0}} \qquad \text{for } x_0 > 0. \tag{21}$$

Similarly

$$\overline{P}(x_0) = \frac{2}{\sqrt{-x_0}} \int_0^{\pi/2} \mathscr{P}(\vartheta)\, (\sec\vartheta)^{\frac{1}{2}}\, d\vartheta = 0 \qquad \text{for } x_0 < 0,$$

since the substitution $\cos\vartheta = \alpha^{\frac{1}{2}}$ leads to the same α-integrals (20) and hence to (21) with $+$ changed to $-$ in the bracket.

⋆ This is the result of applying the transformation

$$F(a, b, c; z) = (1-z)^{-a} F(a, c-b, c; z/(z-1))$$

(Erdélyi *et al.* 1953, p. 64) to the hypergeometric functions in (15).

Appendix 2

The integrand in (19) is singular not only on the circumferences of the ellipses but also for $z = \eta$ when $x \geqq \xi$ (and especially when $x = \xi$). In order to transpose the latter to the boundary of the region of integration, set (Fig. 6a)

$$x = \lambda + r\cos\vartheta, \quad \xi = \lambda - r\cos\vartheta,$$
$$z = \mu + r\sin\vartheta, \quad \eta = \mu - r\sin\vartheta.$$

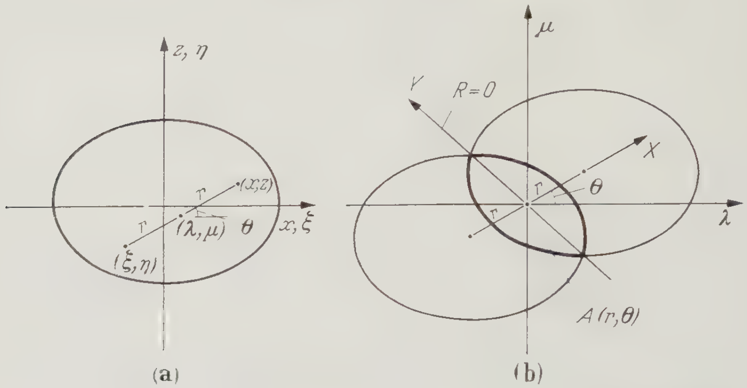

Fig. 6. (a) Change of coordinates x, z, ξ, η to λ, μ, r, ϑ. (b) The integration region $A(r, \vartheta)$ for λ, μ

Then (Fig. 6b) for fixed r, ϑ the point (λ, μ) ranges over the intersection, $A(r, \vartheta)$, of the two ellipses obtained by displacing the original through $(\pm r\cos\vartheta, \pm r\sin\vartheta)$. Thus

$$I = \sqrt{2}\alpha \int_0^{2\pi} \mathscr{P}(\vartheta)\, d\vartheta \int_0^{f^{-\frac{1}{2}}} r^{-\frac{1}{2}}\, dr \iint_{A(r,\vartheta)} \frac{(\lambda^2 - r^2\cos^2\vartheta)\, d\lambda\, d\mu}{\sqrt{R(\lambda, \mu; r, \vartheta)}},$$

where

$$R(\lambda, \mu; r, \vartheta)$$
$$= [1 - (\lambda + r\cos\vartheta)^2 - \alpha^2(\mu + r\sin\vartheta)^2]\,[1 - (\lambda - r\cos\vartheta)^2 - \alpha^2(\mu - r\sin\vartheta)^2]$$

and $f(\vartheta)$ is defined below. By using new coordinates $X = \lambda\cos\vartheta + \alpha^2\mu\sin\vartheta$, $Y = -\alpha\lambda\sin\vartheta + \alpha\mu\cos\vartheta$ and integrating over Y, the integral over A may be reduced to

$$\frac{4}{\alpha} \int_0^{c=0} [B(X; r, \vartheta)\, K(c) - C(X; r, \vartheta)\, E(c)]\, dX$$

where

$$f = \cos^2\vartheta + \alpha^2\sin^2\vartheta, \quad g = \alpha^2\sin^2\vartheta, \quad c = \sqrt{\frac{f - (X + rf)^2}{f - (X - rf)^2}},$$

$$B = \frac{[f - 2g)\, X^2 + 2rfg\, X + f(g - r^2 f^2)]}{f^2 \sqrt{f - (X - rf)^2}}, \quad C = \frac{g}{f^2}\sqrt{f - (X + rf)^2},$$

and K and E are complete elliptic integrals of the first and second kinds respectively.

The limits of integration are fixed for the variables ϑ, s, t where $r = f^{-\frac{1}{2}} s$ and $X = (1-s) f^{\frac{1}{2}} t$. The logarithmic and/or algebraic singularities in the resulting integrand of I at $\vartheta = 0$, $s = 0$, $t = 0$ and $t = 1$ can be removed by setting $\vartheta = \varphi^2$, $s = \sigma^4$, $t = \sin^2 \psi$.

For $\alpha = 0$ there is a further singularity (arising from $f = 0$ at $\vartheta = \pi/2$). However, the ϑ-integration can then be carried out analytically using the results given in Appendix 1, and the remaining σ- and ψ-integrations are regular.

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A Uniqueness Theorem in Magnetohydrodynamics

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Introduction

Consider the flow of an incompressible, viscous, electrically conducting fluid around a finite solid body B, which is assumed to have the same magnetic permeability as the fluid. It is supposed that there are no other boundaries in the fluid, which extends to infinity in three-space. We show that subject to certain conditions on the velocity and magnetic fields $\boldsymbol{v}$ and $\boldsymbol{H}$ and on their first spatial derivatives, the fluid motion and the electromagnetic quantities are uniquely determined by the motion of B and the distributions of $\boldsymbol{v}$ and $\boldsymbol{H}$ at some initial instant, together with the behaviour of the pressure p at large distances from B.

The method used is an extension of that used recently by GRAFFI in a series of papers [*1, 2, 3*] dealing with the corresponding uniqueness theorem for the non-magnetic case, both incompressible and compressible fluids being considered. An important feature of these papers is that uniqueness is proved without assumptions about the convergence at infinity of the velocity, such assumptions being common in earlier papers on this subject (*e.g.* [*4*]). Uniqueness theorems for magnetohydrodynamic flows have been proved by NARDINI [*5*] and KANWAL [*6*] under various hypotheses; FERRARI [*7*], following the method of GRAFFI, has established uniqueness in the magnetohydrodynamic case under conditions of convergence at infinity weaker than those required by NARDINI.

The present result is similar to that of [*7*] as far as conditions at infinity are concerned, but differs in the magnetic boundary conditions used on the surface of the solid B. FERRARI assumes that the tangential component of the magnetic field is specified on the body surface at all times; here all we require is that the magnetic field and the tangential component of the electric field be continuous across the body surface.

Equations of motion

The equations governing the motion of the fluid are, under the usual assumptions of magnetohydrodynamics [*8*],

$$\varrho \frac{\partial \boldsymbol{v}}{\partial t} + \varrho (\boldsymbol{v} \cdot \operatorname{grad}) \boldsymbol{v} - \frac{\mu}{4\pi} (\boldsymbol{H} \cdot \operatorname{grad}) \boldsymbol{H} = -\operatorname{grad} \tilde{\omega} + \varrho \nu \nabla^2 \boldsymbol{v} + \varrho \boldsymbol{X}, \tag{1}$$

$$\frac{\partial \boldsymbol{H}}{\partial t} + (\boldsymbol{v} \cdot \operatorname{grad}) \boldsymbol{H} - (\boldsymbol{H} \cdot \operatorname{grad}) \boldsymbol{v} = \eta \nabla^2 \boldsymbol{H}, \tag{2}$$

$$\operatorname{curl} \boldsymbol{H} = 4\pi\sigma (\boldsymbol{E} + \mu \boldsymbol{v} \times \boldsymbol{H}), \tag{3}$$

$$\operatorname{div} \boldsymbol{v} = 0, \tag{4}$$

$$\operatorname{div} \boldsymbol{H} = 0, \tag{5}$$

where $\boldsymbol{v}$ is the fluid velocity, p the pressure, ϱ the density, ν the kinematical viscosity, $\boldsymbol{H}$ the magnetic field, $\boldsymbol{E}$ the electric field, μ the permeability, σ the fluid conductivity, $\eta=(4\pi\mu\sigma)^{-1}$ the magnetic viscosity of the fluid, $\boldsymbol{X}$ the externally applied force, and $\tilde{\omega}=p+\mu H^2/8\pi$.

Let $\tilde{\sigma}$ be the conductivity of the body B, and $\tilde{\eta}=(4\pi\mu\tilde{\sigma})^{-1}$ its magnetic viscosity. The equations satisfied by the magnetic field in B will be (2), (3) and (5), with σ replaced by $\tilde{\sigma}$ and η by $\tilde{\eta}$; in these equations $\boldsymbol{v}$ will denote the velocity of a typical point of B.

B is assumed to be a bounded closed region of three-space, having a piecewise smooth surface S. The boundary conditions which must be satisfied on S are:

$$\boldsymbol{v},\ \boldsymbol{H},\ \text{and the tangential component of } \boldsymbol{E} \text{ are continuous across } S. \tag{6}$$

Let D be the set of points of three-space exterior to B, and let T be the closed time interval $T=(0, t_0)$, where t_0 (>0) is arbitrary but fixed. To make the initial value problem precise we shall impose the following conditions on the various fields defined interior and exterior to B; in these conditions x_i ($i=1, 2, 3$) will denote rectangular Cartesian coordinates measured from some origin 0 fixed in space, and $r^2=x_i x_i$ (using the tensor summation convention).

(i) The velocity components v_j and their first derivatives with respect to the x_i and to the time t are continuous bounded functions of x_i and t for all points in the product set $D\times T$. The second order space derivatives are continuous for all points in $D\times T$.

(ii) The components H_j of the magnetic field and their first derivatives with respect to the space coordinates and the time are continuous bounded functions of x_i and t for all points in $D\times T$ and for all points in $B\times T$. (H_j is continuous across S by (6), but there may be discontinuities across S in $\partial H_j/\partial x_i$.) The second order space derivatives are continuous for all points in $D\times T$ and for all points in $B\times T$.

(iii) The pressure p is continuous and has continuous first derivatives with respect to the x_i in $D\times T$. At infinity p converges to a given value p_0 such that, uniformly in t, for all $t\in T$,

$$p=p_0+O(r^{-1})$$

for large r.

(iv) The v_j are specified throughout D for $t=0$, and the H_j are specified throughout $D\cup B$ for $t=0$.

(v) The v_j are specified on S (and hence, of course, at every point of B) for every $t\geqq 0$.

(vi) The external force components X_j are specified as functions of x_i and t.

Put simply, we envisage a situation in which the body B moves with prescribed and finite velocity in an unbounded fluid in which the pressure approaches a specified value p_0 at large distances from B; in addition the values of $\boldsymbol{v}$ and $\boldsymbol{H}$ at some initial instant are prescribed throughout all space. The content of this paper is summarised in the following

Theorem. *There can be at most one solution of equations* (1)—(5) (*and of the corresponding equations which hold inside* B) *satisfying the conditions* (6) *and* (i)—(vi).

Proof. Suppose there are two solutions v_j, H_j, p and v_j+u_j, H_j+h_j, $p+p'$ which satisfy equations (1)—(5) and conditions (6), (i)—(vi). Then at points

in the fluid,

$$\varrho\left[\frac{\partial u_j}{\partial t}+u_k\frac{\partial}{\partial x_k}(v_j+u_j)+v_k\frac{\partial u_j}{\partial x_k}\right]-\frac{\mu}{4\pi}\left[h_k\frac{\partial}{\partial x_k}(H_j+h_j)+H_k\frac{\partial h_j}{\partial x_k}\right] = -\frac{\partial\tilde{\omega}'}{\partial x_j}+\varrho\,\nu\frac{\partial^2 u_j}{\partial x_k\,\partial x_k}, \tag{7}$$

$$\frac{\partial h_j}{\partial t}+u_k\frac{\partial}{\partial x_k}(H_j+h_j)+v_k\frac{\partial h_j}{\partial x_k}-h_k\frac{\partial}{\partial x_k}(v_j+u_j)-H_k\frac{\partial u_j}{\partial x_k}=\eta\frac{\partial^2 h_j}{\partial x_k\,\partial x_k}, \tag{8}$$

$$\frac{\partial u_i}{\partial x_i}=0, \tag{9}$$

$$\frac{\partial h_i}{\partial x_i}=0, \tag{10}$$

where $\quad\tilde{\omega}'=p'+(\mu/8\pi)\,(2H_j\,h_j+h_j\,h_j)$.

For points lying inside B the corresponding equations are

$$\frac{\partial h_j}{\partial t}+V_k\frac{\partial h_j}{\partial x_k}-h_k\frac{\partial V_j}{\partial x_k}=\tilde{\eta}\frac{\partial^2 h_j}{\partial x_k\,\partial x_k}, \tag{8'}$$

$$\frac{\partial h_i}{\partial x_i}=0 \tag{10'}$$

$\boldsymbol{V}=(V_j)$ being the (prescribed) velocity of a typical point of B.

Let ω_R be the closed ball of centre 0, radius R, and of surface σ_R. We choose R so large that for all $t\in T$, $B\subset\omega_R$. Let $\omega'_R=\omega_R\cap c\,B$, where $c\,B$ denotes the exterior of B. Then on multiplying (7) by u_j and summing, integration of the resulting expression throughout ω'_R gives, after using Green's theorem and the boundary condition $u_i=0$ on S,

$$\begin{aligned}\int\limits_{\omega'_R}\left[\frac{1}{2}\varrho\frac{\partial}{\partial t}(u_j\,u_j)+\varrho\,\nu\frac{\partial u_j}{\partial x_k}\frac{\partial u_j}{\partial x_k}\right]d\omega_R &\\ =-\varrho\int\limits_{\omega'_R}\left[u_j\,u_k\frac{\partial}{\partial x_k}(v_j+u_j)-\frac{\mu}{4\pi\varrho}u_j\left\{h_k\frac{\partial}{\partial x_k}(H_j+h_j)+H_k\frac{\partial h_j}{\partial x_k}\right\}\right]d\omega_R &\\ +\int\limits_{\sigma_R}\left[\varrho\,\nu\,u_j\frac{\partial u_j}{\partial x_k}n_k-\tilde{\omega}'\,u_j\,n_j\right]d\sigma_R-\frac{1}{2}\varrho\int\limits_{\sigma_R}u_j\,u_j\,v_k\,n_k\,d\sigma_R.&\end{aligned} \tag{11}$$

Here $\boldsymbol{n}=(n_j)$ is the normal drawn outward from ω'_R. Similarly, if (8) and (8′) are multiplied by h_j and summed, then integrated throughout ω'_R and B respectively, addition of the resulting expression gives

$$\begin{aligned}\int\limits_{\omega'_R}\left[\frac{1}{2}\frac{\partial}{\partial t}(h_j\,h_j)+\eta\frac{\partial h_j}{\partial x_k}\frac{\partial h_j}{\partial x_k}\right]d\omega_R+\int\limits_{B}\left[\frac{1}{2}\frac{\partial}{\partial t}(h_j\,h_j)+\tilde{\eta}\frac{\partial h_j}{\partial x_k}\frac{\partial h_j}{\partial x_k}\right]d\,B &\\ =\eta\int\limits_{\sigma_R}h_j\frac{\partial h_j}{\partial x_k}n_k\,d\sigma_R+\int\limits_{S}h_j\,\Delta\left(\eta\frac{\partial h_j}{\partial x_k}\right)n_k\,d\,S\,- &\\ -\int\limits_{\omega'_R}h_j\left[u_k\frac{\partial}{\partial x_k}(H_j+h_j)+v_k\frac{\partial h_j}{\partial x_k}-h_k\frac{\partial}{\partial x_k}(v_j+u_j)-H_k\frac{\partial u_j}{\partial x_k}\right]d\omega_R- &\\ -\int\limits_{B}h_j\left(V_k\frac{\partial h_j}{\partial x_k}-h_k\frac{\partial V_j}{\partial x_k}\right)d\,B.&\end{aligned} \tag{12}$$

In obtaining this the boundary condition (6) on S has been used; $\Delta(f)$ is the discontinuity in f on crossing S. Combining (11) and (12),

$$\begin{aligned}
&\int_{\omega'_R}\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho u_j u_j+\frac{\mu}{8\pi}h_j h_j\right)+\varrho\nu\frac{\partial u_j}{\partial x_k}\frac{\partial u_j}{\partial x_k}+\frac{\eta\mu}{4\pi}\frac{\partial h_j}{\partial x_k}\frac{\partial h_j}{\partial x_k}\right]d\omega_R+\\
&\quad+\int_B\frac{\mu}{4\pi}\left[\frac{\partial}{\partial t}\left(\frac{1}{2}h_j h_j\right)+\tilde{\eta}\frac{\partial h_j}{\partial x_k}\frac{\partial h_j}{\partial x_k}\right]dB\\
&=-\varrho\int_{\omega'_R}u_j u_k\frac{\partial}{\partial x_k}(v_j+u_j)\,d\omega_R-\frac{1}{2}\varrho\int_{\sigma_R}u_j u_j v_k n_k\,d\sigma_R-\\
&\quad-\int_{\sigma_R}\left[\tilde{\omega}' u_j n_j-\varrho\nu u_j\frac{\partial u_j}{\partial x_k}n_k\right]d\sigma_R+\\
&\quad+\frac{\mu}{4\pi}\int_{\omega'_R}\left[u_j\left\{h_k\frac{\partial}{\partial x_k}(H_j+h_j)+H_k\frac{\partial h_j}{\partial x_k}\right\}+\right.\\
&\quad\left.+h_j\left\{h_k\frac{\partial}{\partial x_k}(v_j+u_j)+H_k\frac{\partial u_j}{\partial x_k}-u_k\frac{\partial}{\partial x_k}(H_j+h_j)-v_k\frac{\partial h_j}{\partial x_k}\right\}\right]d\omega_R-\\
&\quad-\frac{\mu}{4\pi}\int_B h_j\left(V_k\frac{\partial h_j}{\partial x_k}-h_k\frac{\partial V_j}{\partial x_k}\right)dB+\\
&\quad+\frac{\eta\mu}{4\pi}\int_{\sigma_R}h_j\frac{\partial h_j}{\partial x_k}n_k\,d\sigma_R+\frac{\mu}{4\pi}\int_S h_j\,\Delta\left(\eta\frac{\partial h_j}{\partial x_k}\right)n_k\,dS.
\end{aligned}\tag{13}$$

Now

$$\int_S h_j\,\Delta\left(\eta\frac{\partial h_j}{\partial x_k}\right)n_k\,dS=\int_S \boldsymbol{h}\cdot\Delta[\eta\operatorname{curl}\boldsymbol{h}\times\boldsymbol{n}]\,dS+\int_S h_j\,\Delta\left(\eta\frac{\partial h_k}{\partial x_j}\right)n_k\,dS.$$

Denoting the electric fields in the two postulated possible flows by $\boldsymbol{E}$ and $\boldsymbol{E}+\boldsymbol{E}'$, we have, in the fluid,

$$\boldsymbol{E}=-\mu\,\boldsymbol{v}\times\boldsymbol{H}+(4\pi\sigma)^{-1}\operatorname{curl}\boldsymbol{H}$$

$$\boldsymbol{E}+\boldsymbol{E}'=-\mu(\boldsymbol{v}+\boldsymbol{u})\times(\boldsymbol{H}+\boldsymbol{h})+(4\pi\sigma)^{-1}\operatorname{curl}(\boldsymbol{H}+\boldsymbol{h})$$

so that

$$\boldsymbol{E}'=(4\pi\sigma)^{-1}\operatorname{curl}\boldsymbol{h}-\mu\{\boldsymbol{u}\times(\boldsymbol{H}+\boldsymbol{h})+(\boldsymbol{v}\times\boldsymbol{h})\}.$$

For points inside B a similar expression for $\boldsymbol{E}'$ exists, with σ replaced by $\tilde{\sigma}$ and $\boldsymbol{v}$ by $\boldsymbol{V}$. At a solid-fluid interface the tangential components of the electric field must be continuous, and this provides the condition

$$0=\boldsymbol{n}\times\Delta(\boldsymbol{E}')=\mu\,\boldsymbol{n}\times\Delta[\eta\operatorname{curl}\boldsymbol{h}-\boldsymbol{u}\times(\boldsymbol{H}+\boldsymbol{h})-(\boldsymbol{v}\times\boldsymbol{h})].$$

Since $\boldsymbol{v}$ and $\boldsymbol{h}$ are continuous across S, and $\boldsymbol{u}=0$ on S, this reduces to

$$\Delta[\eta\operatorname{curl}\boldsymbol{h}]\times\boldsymbol{n}=0.$$

Hence

$$\begin{aligned}
\int_S h_j\,\Delta\left(\eta\frac{\partial h_j}{\partial x_k}\right)n_k\,dS&=\int_S h_j\,\Delta\left(\eta\frac{\partial h_k}{\partial x_j}\right)n_k\,dS\\
&=-\eta\int_{\sigma_R}h_j\frac{\partial h_k}{\partial x_j}n_k\,d\sigma_R+\eta\int_{\omega'_R}\left[\frac{\partial h_j}{\partial x_k}\frac{\partial h_j}{\partial x_k}-(\operatorname{curl}\boldsymbol{h})^2\right]d\omega_R+\\
&\quad+\tilde{\eta}\int_B\left[\frac{\partial h_j}{\partial x_k}\frac{\partial h_j}{\partial x_k}-(\operatorname{curl}\boldsymbol{h})^2\right]dB
\end{aligned}$$

using Green's theorem. Equation (13) thus becomes

$$\begin{aligned}
&\int\limits_{\omega'_R}\left[\frac{1}{2}\frac{\partial}{\partial t}\left(\varrho\, u_j u_j + \frac{\mu}{4\pi} h_j h_j\right) + \varrho\,\nu\,\frac{\partial u_j}{\partial x_k}\frac{\partial u_j}{\partial x_k}\right] d\omega_R + \frac{\mu}{4\pi}\int\limits_B \frac{1}{2}\frac{\partial}{\partial t}(h_j h_j)\, dB \\
&\quad = -\varrho\int\limits_{\omega'_R} u_j u_k \frac{\partial}{\partial x_k}(v_j + u_j)\, d\omega_R - \frac{1}{2}\varrho\int\limits_{\sigma_R} u_j u_j v_k n_k\, d\sigma_R - \\
&\qquad - \int\limits_{\sigma_R}\left(\tilde{\omega}' u_j n_j - \varrho\,\nu\, u_j \frac{\partial u_j}{\partial x_k} n_k\right) d\sigma_R + \\
&\qquad + \frac{\mu}{4\pi}\int\limits_{\omega'_R}\left[(u_j h_k - h_j u_k)\frac{\partial}{\partial x_k}(H_j + h_j) + h_j h_k \frac{\partial}{\partial x_k}(v_j + u_j)\right] d\omega_R + \qquad (14) \\
&\qquad + \frac{\mu}{4\pi}\int\limits_{\sigma_R}\left(H_k u_j h_j - \frac{1}{2} h_j h_j v_k\right) n_k\, d\sigma_R + \frac{\mu}{4\pi}\int\limits_B h_j h_k \frac{\partial V_j}{\partial x_k}\, dB + \\
&\qquad + \frac{\eta\mu}{4\pi}\int\limits_{\sigma_R} h_j\left(\frac{\partial h_j}{\partial x_k} - \frac{\partial h_k}{\partial x_j}\right) n_k\, d\sigma_R - \\
&\qquad - \frac{\eta\mu}{4\pi}\int\limits_{\omega'_R} (\operatorname{curl} \boldsymbol{h})^2\, d\omega_R - \frac{\tilde{\eta}\mu}{4\pi}\int\limits_B (\operatorname{curl} \boldsymbol{h})^2\, dB.
\end{aligned}$$

Integrating (14) with respect to t from 0 to t_1, and then again with respect to t_1 from 0 to b ($\leqq t_0$), where b is a strictly positive constant to be specified later, there follows

$$\begin{aligned}
0 &\leqq \int_0^b dt_1 \int\limits_{\omega'_R} \frac{1}{2}\varrho\left(u_j u_j + \frac{\mu}{4\pi} h_j h_j\right) d\omega_R + \\
&\quad + \varrho\,\nu\int_0^b dt_1\int_0^{t_1} dt \int\limits_{\omega'_R} \frac{\partial u_j}{\partial x_k}\frac{\partial u_j}{\partial x_k}\, d\omega_R + \int_0^b dt_1 \int\limits_B \frac{\mu}{8\pi} h_j h_j\, dB + \\
&\quad + \frac{\eta\mu}{4\pi}\int_0^b dt_1\int_0^{t_1} dt\int\limits_{\omega'_R} (\operatorname{curl}\boldsymbol{h})^2\, d\omega_R + \frac{\tilde{\eta}\mu}{4\pi}\int_0^b dt_1\int_0^{t_1} dt \int\limits_B (\operatorname{curl}\boldsymbol{h})^2\, dB \\
&= -\varrho\int_0^b dt_1\int_0^{t_1} dt\int\limits_{\omega'_R} u_j u_k \frac{\partial}{\partial x_k}(v_j + u_j)\, d\omega_R - \frac{1}{2}\varrho\int_0^b dt_1 \int_0^{t_1} dt \int\limits_{\sigma_R} u_j u_j v_k n_k\, d\sigma_R - \\
&\quad - \int_0^b dt_1 \int_0^{t_1} dt \int\limits_{\sigma_R}\left(\tilde{\omega}' u_j n_j - \varrho\,\nu\, u_j\frac{\partial u_j}{\partial x_k} n_k\right) d\sigma_R + \qquad (15) \\
&\quad + \frac{\mu}{4\pi}\int_0^b dt_1\int_0^{t_1} dt\int\limits_{\omega'_R}\left[(u_j h_k - h_j u_k)\frac{\partial}{\partial x_k}(H_j + h_j) + h_j h_k\frac{\partial}{\partial x_k}(v_j + u_j)\right] d\omega_R + \\
&\quad + \frac{\mu}{4\pi}\int_0^b dt_1 \int_0^{t_1} dt\int\limits_{\sigma_R}\left(H_k u_j h_j - \frac{1}{2} h_j h_j v_k\right) n_k\, d\sigma_R + \\
&\quad + \frac{\mu}{4\pi}\int_0^b dt_1\int_0^{t_1} dt\int\limits_B h_j h_k \frac{\partial V_j}{\partial x_k}\, dB + \frac{\eta\mu}{4\pi}\int_0^b dt_1 \int_0^{t_1} dt \int\limits_{\sigma_R} h_j\left(\frac{\partial h_j}{\partial x_k} - \frac{\partial h_k}{\partial x_j}\right) n_k\, d\sigma_R
\end{aligned}$$

use being made of the initial condition (iv).

Let

$$N = \sup\left|\frac{\partial}{\partial x_k}(v_j + u_j)\right|, \qquad N' = \sup|v_k|, \qquad Q = \sup|u_j|,$$

$$M = \sup\left|\frac{\partial}{\partial x_k}(H_j + h_j)\right|, \quad M' = \sup|H_k|$$

the suprema being taken over all points in $D \times T$ and over all j, k; let

$$N'' = \sup\left|\frac{\partial V_j}{\partial x_k}\right|$$

the supremum being taken over all points in $B \times T$ and over all j, k. Then using Cauchy's inequality repeatedly, we find that for points in $D \times T$,

$$\left|u_j u_k \frac{\partial}{\partial x_k}(v_j + u_j)\right| \leqq 3 N u_j u_j$$

$$|u_j u_j v_k n_k| \leqq 3 N' u_j u_j$$

$$\left|u_j \frac{\partial u_j}{\partial x_k} n_k\right| \leqq \frac{3}{2} u_j u_j + \frac{1}{2} \frac{\partial u_j}{\partial x_k} \frac{\partial u_j}{\partial x_k}$$

$$\left|u_j h_k \frac{\partial}{\partial x_k}(H_j + h_j)\right| \leqq \frac{3}{2} M(u_j u_j + h_j h_j)$$

$$\left|h_j h_k \frac{\partial}{\partial x_k}(v_j + u_j)\right| \leqq 3 N h_j h_j$$

$$|H_k u_j h_j n_k| \leqq \frac{3}{2} M'(u_j u_j + h_j h_j)$$

$$|h_j h_j v_k n_k| \leqq 3 N' h_j h_j$$

$$\left|h_j\left(\frac{\partial h_j}{\partial x_k} - \frac{\partial h_k}{\partial x_j}\right) n_k\right| \leqq \frac{3}{2} h_j h_j + (\operatorname{curl} \boldsymbol{h})^2$$

$$|h_k h_k u_j n_j| \leqq 3 Q h_j h_j$$

while for points in $B \times T$,

$$\left|h_j h_k \frac{\partial V_j}{\partial x_k}\right| \leqq 3 N'' h_j h_j.$$

Finally we need an estimate for $\int_{\sigma_R} p' u_j n_j \, d\sigma_R$. Using Schwarz's inequality,

$$\left|\int_{\sigma_R} p' u_j n_j \, d\sigma_R\right| \leqq \left[\int_{\sigma_R} p'^2 \, d\sigma_R \cdot \int_{\sigma_R} u_j u_j \, d\sigma_R\right]^{\frac{1}{2}}$$

and since $p' = O(r^{-1})$ by condition (iii), it follows that there exists a strictly positive constant c such that

$$\left|\int_{\sigma_R} p' u_j n_j \, d\sigma_R\right| \leqq c\left[\int_{\sigma_R} u_j u_j \, d\sigma_R\right]^{\frac{1}{2}}$$

provided R is large enough.

Using all these inequalities in (15), we find

$$
\begin{aligned}
0 \leqq G(R) \leqq 3b\left(N\varrho + \frac{M\mu}{4\pi}\right)\int_0^b dt_1 \int_{\omega'_R} u_j u_j \, d\omega_R + \\
+ \frac{3}{2} b\left(N'\varrho + \varrho\nu + \frac{M'\mu}{2\pi}\right)\int_0^b dt_1 \int_{\sigma_R} u_j u_j \, d\sigma_R + \frac{1}{2}\varrho\nu \int_0^b dt_1 \int_0^{t_1} dt \int_{\sigma_R} \frac{\partial u_j}{\partial x_k}\,\frac{\partial u_j}{\partial x_k}\, d\sigma_R + \\
+ \frac{3\mu}{4\pi} b(M+N)\int_0^b dt_1 \int_{\omega'_R} h_j h_j \, d\omega_R + \frac{3\mu}{8\pi} b(2M'+N'+\eta-Q)\int_0^b dt_1 \int_{\sigma_R} h_j h_j \, d\sigma_R + \\
+ \frac{3\mu}{4\pi} b N'' \int_0^b dt_1 \int_B h_j h_j \, dB + \frac{\mu}{4\pi}\eta \int_0^b dt_1 \int_0^{t_1} dt \int_{\sigma_R} (\operatorname{curl} \boldsymbol{h})^2 \, d\sigma_R + \\
+ c\int_0^b dt_1 \int_0^{t_1} dt \left[\int_{\sigma_R} u_j u_j \, d\sigma_R\right]^{\frac{1}{2}}
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
G(R) = \int_0^b dt_1 \int_{\omega'_R} \frac{1}{2}\left(\varrho u_j u_j + \frac{\mu}{4\pi} h_j h_j\right) d\omega_R - \int_0^b dt_1 \int_B \frac{\mu}{8\pi} h_j h_j \, dB - \\
+ \varrho\nu \int_0^b dt_1 \int_0^{t_1} dt \int_{\omega'_R} \frac{\partial u_j}{\partial x_k}\,\frac{\partial u_j}{\partial x_k}\, d\omega_R + \\
+ \frac{\eta\mu}{4\pi}\int_0^b dt_1 \int_0^{t_1} dt \int_{\omega'_R} (\operatorname{curl} \boldsymbol{h})^2 \, d\omega_R + \frac{\tilde{\eta}\mu}{4\pi}\int_0^b dt_1 \int_0^{t_1} dt \int_B (\operatorname{curl} \boldsymbol{h})^2 \, dB.
\end{aligned}
\tag{17}
$$

By Schwarz's inequality,

$$
\begin{aligned}
\int_0^b dt_1 \int_0^{t_1} dt \left[\int_{\sigma_R} u_j u_j \, d\sigma_R\right]^{\frac{1}{2}} &\leqq \int_0^b dt_1 \left[\left\{\int_0^{t_1} dt\right\}\left\{\int_0^{t_1} dt \int_{\sigma_R} u_j u_j \, d\sigma_R\right\}\right]^{\frac{1}{2}} \\
&\leqq b\left[\left\{\int_0^b dt\right\}\left\{\int_0^b dt \int_{\sigma_R} u_j u_j \, d\sigma_R\right\}\right]^{\frac{1}{2}} = b^{\frac{3}{2}}\left[\int_0^b dt \int_{\sigma_R} u_j u_j \, d\sigma_R\right]^{\frac{1}{2}}.
\end{aligned}
$$

Putting

$$
m_1 = \max\left[6N + \frac{3M\mu}{2\pi\varrho},\ 6(M+N),\ 6N''\right]
$$

and choosing $b = \max\{t_0/n : t_0/n \leqq (2m_1)^{-1},\ n \text{ integral}\}$, (17) becomes

$$
\begin{aligned}
G(R) \leqq \frac{1}{2} G(R) + \frac{3}{2} b\left(N'\varrho + \nu\varrho + \frac{M'\mu}{2\pi}\right)\int_0^b dt_1 \int_{\sigma_R} u_j u_j \, d\sigma_R + \\
+ \frac{1}{2}\varrho\nu \int_0^b dt_1 \int_0^{t_1} dt \int_{\sigma_R} \frac{\partial u_j}{\partial x_k}\,\frac{\partial u_j}{\partial x_k}\, d\sigma_R + \frac{3\mu}{8\pi} b(2M'+N'+\eta+Q)\int_0^b dt_1 \int_{\sigma_R} h_j h_j \, d\sigma_R + \\
+ \frac{\eta\mu}{4\pi}\int_0^b dt_1 \int_0^{t_1} dt \int_{\sigma_R} (\operatorname{curl} \boldsymbol{h})^2 \, d\sigma_R + c\, b^{\frac{3}{2}}\left[\int_0^b dt_1 \int_{\sigma_R} u_j u_j \, d\sigma_R\right]^{\frac{1}{2}}.
\end{aligned}
$$

On setting

$$m = 2\max\left[3b\left(N' + \nu + \frac{M'\mu}{2\pi\varrho}\right), \frac{1}{2}, 3b(2M' + N' + \eta + Q)\right]$$

we find

$$G(R) \leqq m\,G'(R) + l[G'(R)]^{\frac{1}{2}} \tag{18}$$

where $l = 2c\,b^{\frac{3}{2}}(2/\varrho)^{\frac{1}{2}}$, $G'(R) \equiv dG/dR$.

We propose next to show that $G(R) \equiv 0$; the argument is exactly the same as that given by GRAFFI [2], but we reproduce it here for the sake of completeness.

In view of conditions (i) and (ii),

$$G(R) = O(R^3) \tag{19}$$

for large R. Suppose there exists an R_0 such that $G(R_0) \neq 0$. Since $G(R)$ is always non-negative, this implies that $G(R_0) > 0$. Because $G'(R)$ is also non-negative, $G(R)$ is monotonic increasing, and hence

$$0 < G(R_0) \leqq G(R) \leqq m\,G'(R) + l[G'(R)]^{\frac{1}{2}} \tag{20}$$

for all $R > R_0$. It follows that there is a constant $\alpha > 0$ such that $[G'(R)]^{\frac{1}{2}} \geqq \alpha$ for all $R > R_0$. Then (20) may be written

$$G(R) \leqq \left[m + \frac{l}{[G'(R)]^{\frac{1}{2}}}\right] G'(R) \leqq \left(m + \frac{l}{\alpha}\right) G'(R)$$

that is,

$$\frac{G'(R)}{G(R)} \geqq \left(m + \frac{l}{\alpha}\right)^{-1} = \delta > 0$$

and so

$$G(R) \geqq G(R_0)\exp\{\delta(R - R_0)\}$$

for all $R > R_0$. This contradicts (19), and hence $G(R) \equiv 0$. We deduce that $u_j = 0$ throughout $D \times (0, b)$, and $h_j = 0$ throughout $(D \cup B) \times (0, b)$.

Now integrate (14) with respect to t from b to $b + t_1$, and then with respect to t_1 from 0 to b. The previous arguments may then be repeated to show that $u_j = 0$ throughout $D \times (b, 2b)$, and that $h_j = 0$ throughout $(D \cup B) \times (b, 2b)$. Proceeding in this way we eventually cover the whole of the arbitrary interval $T = (0, t_0)$ in steps of length b, so that we have finally

$$u_j = 0 \quad \text{in} \quad D \times T, \qquad h_j = 0 \quad \text{in} \quad (D \cup B) \times T.$$

It follows easily from (7) and (iii) that $p' = 0$ in $D \times T$, and hence the solution is unique for all $t \in T$. But T is arbitrary, and hence the theorem is proved.

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